DUKE UNIV. LIBRARY
DURHAM, N. C.
AUG 26 1938

VOLUME 24 AUGUST, 1938 NUMBER 8

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

OF THE

UNITED STATES OF AMERICA

EXECUTIVE COMMITTEE

EDWIN BIDWELL WILSON, *Managing Editor*

RAYMOND PEARL LUDVIG HEKTOEN

PUBLISHED MONTHLY

Publication Office: Mack Printing Company, Easton, Pa.

Editorial Office: Harvard School of Public Health, Boston 17

Home Office of the Academy: Washington, D. C.

Entered as second-class matter June 21, 1920, at the post-office at Easton, Pennsylvania, under the act of August 24, 1912. Accepted for mailing at a special rate of postage, provided for in section 1103, act of October 3, 1917. Authorized June 18, 1920.

Price per annum, $5.00

OFFICERS OF THE ACADEMY

PRESIDENT ..... F. R. LILLIE
VICE PRESIDENT ..... A. L. DAY
FOREIGN SECRETARY ..... L. J. HENDERSON
HOME SECRETARY ..... FRED. E. WRIGHT
TREASURER ..... ARTHUR KEITH

EDITORIAL BOARD

The HOME SECRETARY and the FOREIGN SECRETARY of the ACADEMY
The CHAIRMAN OF THE NATIONAL RESEARCH COUNCIL

V. BUSH	L. J. HENDERSON	W. R. MILES
R. E. COKER	W. S. HUNTER	R. A. MILLIKAN
A. L. DAY	F. B. JEWETT	H. R. MOODY
L. P. EISENHART	A. B. LAMB	W. J. V. OSTERHOUT
R. G. HARRISON	E. R. LONG	F. K. RICHTMYER
L. HEKTOEN	C. R. LONGWELL	F. H. SEARES

INFORMATION TO SUBSCRIBERS

SUBSCRIPTIONS TO THE PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES at the rate of $5.00 per annum should be made payable to the National Academy of Sciences, and sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C. Single numbers, $0.50.

PAST publications of the NATIONAL ACADEMY OF SCIENCES from 1863 to 1926 and of the NATIONAL RESEARCH COUNCIL from 1916 to 1925 are listed in the Index to the First Ten Volumes of these PROCEEDINGS. This Index appeared as Part 2 of Volume 13, No. 1, January, 1927, and is for sale separately at the price of $1.50. Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

PAST volumes of the PROCEEDINGS may be obtained at five dollars per volume unless the sale of the volume would break a complete set of Volumes I to X. Single numbers may be obtained for fifty cents except where the sale of such numbers would break up a complete volume. Only two hundred complete sets of the PROCEEDINGS are available for sale—Volume I to X, price $75.00 (including Index, see *supra*). Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

The following publications are issued by the NATIONAL RESEARCH COUNCIL. Orders and inquiries should be addressed: Publication Office, National Research Council, Constitution Ave. & 21st Street, Washington, D. C.

(A) THE BULLETIN OF THE NATIONAL RESEARCH COUNCIL published at irregular intervals. Individual numbers priced variously. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

(B) THE REPRINT AND CIRCULAR SERIES OF THE NATIONAL RESEARCH COUNCIL, individual numbers variously priced. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 24 August 15, 1938 Number 8
Copyright 1938 by The National Academy of Sciences

THE ORIGIN OF MAIZE

By P. C. Mangelsdorf and R. G. Reeves

Texas Agricultural Experiment Station and A. & M. College of Texas, College Station, Texas

Communicated July 5, 1938

To botanists and archaeologists alike, the origin of Indian corn or maize has been a most intriguing problem, for maize was the basic food plant of the ancient American civilizations. Both DeCandolle[1] and Darwin[2] concluded that maize had its origin in America and the possibility of a pre-Columbian distribution in China has since been dismissed by Laufer's comprehensive studies.[3]

The wild form of maize is unknown and perhaps is now extinct. No fossil remains are known, the one supposedly fossil ear[4] having recently been shown to be an artifact.[5] In the absence of evidence from these and other sources there is left only one recourse in an attack on the problem, a study of the plant itself and of its relatives. Since evolution is primarily a matter of gene change and chromatin rearrangement, we have assumed that cytogenetic investigations of the hybrids of maize and its relatives would provide the most promising avenue of attack upon this problem.

The tribe Maydeae to which maize is assigned comprises eight genera, of which only three, Zea, Euchlaena and Tripsacum, are American. The relationship of the Oriental genera to maize is remote, and they appear to have no immediate bearing on the problem of the origin of maize. Euchlaena, *E. mexicana* Schrad., or teosinte, is an annual confined to Mexico and western Guatemala. In Mexico, it occurs as a weed in and around the corn fields, but in Guatemala, in at least one locality, it is found as a truly wild species.[6] A perennial, tetraploid form, *E. perennis* Hitchc., is also known but it occurs only in a single colony in Mexico and is of questionable specific rank. Tripsacum, a perennial comprising six or seven species, is widely distributed from Central America through the southern and eastern states, and is occasionally found in parts of South America.

Three general theories with regard to the origin of maize have had a vogue: (1) That it originated from pod-corn, which differs from normal maize by a single dominant gene governing the development of a brittle,

readily disarticulating rachis and the production of prominent glumes enclosing the seeds (cf. Saint-Hilaire,[7] Sturtevant[8]). (2) That maize originated from teosinte by direct selection, by large scale mutations or by the hybridization of Euchlaena with some unknown grass (cf. Harshberger,[9] Collins[10]). (3) That maize, teosinte and Tripsacum have descended along independent lines from a remote common ancestor (cf. Weatherwax[11]).

The rather general acceptance of the theory that Euchlaena has played some rôle in the ancestry of maize has led to the assumptions that maize had its origin in Central America or Mexico, and that agriculture, which preceded all advanced cultures and civilizations in the New World, also had its origin there (cf. Spinden[12]).

Our cytogenetic studies covering an eight-year period have led to the surprising conclusion that Euchlaena, far from being the progenitor of maize, is the descendant of a natural hybrid of Zea and Tripsacum.

Our experiments began with hybrids of Zea and Euchlaena made to determine how the genes which differentiate the two genera are distributed on the chromosomes. Most orthodox geneticists assume that differences between genera and species are genic. Hybrids of Zea and Euchlaena furnish excellent material, not only for testing this assumption but also for determining how the genes which differentiate the genera are distributed on the chromosomes. Zea and Euchlaena, differing widely in many quantitative characteristics, have the same chromosome number, 10, hybridize readily, producing highly fertile hybrids, where chromosome pairing is almost complete, and where crossing-over occurs almost as readily as in pure Zea.[13] Finally, there are available for each of the ten chromosomes of maize two or more well-known genes which may be used as markers.

The procedure was to cross Euchlaena with stocks of maize in which certain chromosomes were each marked by two genes, and then to back-cross the F_1 hybrid to a double recessive maize stock. In the succeeding generation of each test four classes of plants appeared, two of which were cross-over classes, the other two parental combinations. Plants which exhibited both of the marker genes from Zea had presumably inherited intact, except for rare double cross-overs, the section of chromatin lying between these two genes, while plants which exhibited the two alleles from Euchlaena had received intact a corresponding section from Euchlaena. As these two groups of plants were, on the average, alike in all other genes and in their cytoplasm, a comparison of the two groups in various quantitative characteristics by which the two parental genera differ, is essentially a comparison of two corresponding regions of the chromatin in the two genera.

Tests of this nature have been completed with only four of the ten chromosomes; but these show (1) that the differences between Zea and Euchlaena are genic; (2) that the genes which differentiate the two genera are not distributed at random on all chromosomes. The genes on chromosome No. 2

marked by *B* and *lg*, 34 units apart, on chromosome No. 6 marked by *Y* and *Pl*, 28 units apart and on chromosome No. 9 marked by *wx* which occupies a region approximately 45 units in length which does not cross-over regularly with the Euchlaena homolog,[14] have but little effect upon the quantitative characteristics which differentiate Zea and Euchlaena. But the genes on chromosome No. 4 marked by *su* and *Tu*, 29 units apart, have a decided effect upon almost all the quantitative characteristics by which the two genera were distinguished in this study.

This non-randomness becomes explicable only after the results of hybridizing Zea and Tripsacum have been considered. Hybridization of these two genera is rather difficult but can be accomplished by the use of a simple technique which involves shortening the styles or silks of maize.[15] With this procedure hybrid seeds are produced in abundance, and though most of these abort at an early stage, a few reach maturity and are capable of germination.

The F_1 hybrids are quite vigorous but completely pollen-sterile. Studies of meiosis show very little pairing between the 10 Zea and the 18 Tripsacum chromosomes although a feeble association is occasionally observed. Unreduced gametes are regularly formed and are apparently the only ones which function. The hybrid sets a small percentage of seeds when pollinated with either of the parental genera or with the third genus, Euchlaena. As the backcross to Zea is the only one which has a direct bearing on the origin of maize, it is the one which will be considered here.

This triploid hybrid possesses 38 chromosomes of which 20 are derived from Zea; 18 from Tripsacum. At meiosis the 20 Zea chromosomes pair completely to form 10 bivalents which behave quite regularly. The 18 Tripsacum chromosomes remain unpaired except that there is an occasional feeble association of one or severalTripsacum chromosomes with the Zea bivalents. The triploid hybrid is completely pollen-sterile but exhibits a fertility of 21 per cent when pollinated by Zea.

In the succeeding generation all plants possess 20 Zea chromosomes plus (usually) one or more Tripsacum chromosomes. The distribution of 178 plants in this population, with respect to the number of Tripsacum chromosomes is as follows: 0;25, 1;76, 2;22, 3;4, 4;8. 5;3, 6;3, 7;4, 8;6, 9;5, 10 or more; 22. Plants with no extra Tripsacum chromosomes are usually typical maize plants with no evidence of Tripsacum influence. A few $2n$ plants, however, exhibit a slight trace of Tripsacum influence, indicating the presence of a few Tripsacum genes. As the number of Tripsacum chromosomes increases, the Tripsacum influence becomes more perceptible; but the relationship is not linear, since one or two Tripsacum chromosomes have a much greater relative effect than a larger number.

It had been planned to isolate from this population 18 different $2n + 1$ stocks in which each of the 18 Tripsacum chromosomes was represented

once, so that the effects of each of the Tripsacum chromosomes could be studied separately, but as the Tripsacum chromosomes were never transmitted through the pollen, and as there was a marked selective action against their transmission through the ovules, many of the stocks were lost. Rather comprehensive studies were completed, however, in one of the $2n + 1$ stocks in which the Tripsacum chromosome was marked by an allele of the *su* gene from maize chromosome No. 4. Prophase studies of the $2n + 1$ plants segregating from this stock showed that the Tripsacum chromosome paired most frequently with Zea chromosome No. 1 but it was also observed pairing with Nos. 4, 5 and 10.

In a stock that is homozygous for the maize gene *su*, the Tripsacum chromosome bearing the allele of *su* is identified by starchiness of the seeds. Starchy seeds when planted give rise to $2n + 1$ plants; sugary seeds from the same ears produce normal $2n$ maize plants. Exceptions to this rule have shown that there is sometimes an exchange of chromatin between the Tripsacum chromosome and one or more of the maize chromosomes. We have had $2n$ plants which exhibited a definite Tripsacum influence including the allele of *su*, and $2n + 1$ plants with irregularities in addition to those attributable to the extra chromosome.

The evidence that there can be an interchange of Zea and Tripsacum chromatin has led to a consideration of the possible consequences of a natural hybridization of Zea and Tripsacum. Obviously there would have been a rapid elimination of the unbalanced types and a quick return to the two parental types; pure Tripsacum, or Tripsacum with a few Zea genes; pure Zea or Zea with a few Tripsacum genes. The infected Tripsacum might have been slightly less able to survive in nature than pure Tripsacum; the infected Zea would have been better fitted to survive in nature only if the Tripsacum infection had provided certain characteristics which maize now lacks, particularly a means of dispersal and protection for the seed, characteristics which Tripsacum possesses in its prominent, horny glumes and brittle rachis.

A description of a maize plant possessing these two Tripsacum characteristics is obviously a rough description of a plant already in existence, Euchlaena; consequently we began to consider seriously for the first time an unpublished suggestion made by Dr. Edgar Anderson, that Euchlaena might be the product of the natural hybridization of Zea and Tripsacum.

A consideration of this hypothesis reveals an amazing amount of evidence in its support, of which the most obvious is the fact that Euchlaena is intermediate between, or identical with, its two putative parents in all of its characteristics. It does not possess, apparently, a single characteristic which might not have been received from either Tripsacum or Zea. If Euchlaena represents the end-product of an independent line of descent from a remote common ancestor, it is almost inconceivable that it should

not have acquired some distinctive characteristics in which it differed from both Zea and Tripsacum.

Convincing evidence of the hybrid nature of Euchlaena has come from a reëxamination of the segregates from hybrids of Zea and Euchlaena which has revealed that Euchlaena differs genetically from Zea primarily by four segments of chromatin which bear genes with Tripsacum effects. The material involved in these studies is the same which had previously been used to study linkage between quantitative and qualitative characters, as reported earlier in this paper. The previous studies had been made on the assumption that Zea and Euchlaena are distinct genera and that the numerous quantitative differences between them are generic differences. Consequently we had measured as many as possible of the characteristics in which the two genera differed. In reapproaching the problem we attempted to ignore all differences of the kind which occur within Zea itself and to confine ourselves to the essential characteristics which distinguish Euchlaena from Zea. These are only four: prominent horny glumes, a brittle rachis, unpaired pistillate spikelets and a distichous arrangement of the spikelets of the pistillate inflorescence. As Zea is dominant for paired spikelets and the F_1 hybrid had been back-crossed to Zea, it was impossible to study the segregation for this character in this material. In attempting to arrive at a classification with respect to the remaining characteristics, numerous difficulties were encountered, but it finally became apparent that segregates possessing none of these characteristics and hence essentially typical maize plants were occurring in a ratio of approximately 1 in 16 while segregates possessing all of these characteristics and duplicating the F_1 hybrids were occurring in approximately the same ratio. This frequent reappearance of the parental types indicated that only a relatively small number of factors could be involved and the 15:1 ratio suggested a four-factor segregation which in a back-cross should result in five distinct classes occurring in a ratio of 1:4:6:4:1. It was found that the segregates could be readily classified according to such a scheme, the actual numbers of plants falling into the different classes being 22, 190, 384, 303 and 21. Although the distribution differs significantly from the theoretical ratio, there is little doubt that the classification is essentially correct, for the deviations are largely accounted for by several disturbing influences. Furthermore, each class can be divided into its component sub-classes, and it has been possible to identify the 16 distinct classes expected from a four-factor segregation.

Although the population segregated in a four-factor ratio, it is scarcely conceivable that the differences between Zea and Euchlaena are due alone to four genes. Certain exceptional plants which exhibit part, but not all, of the effects of one of these factors indicate that each factor represents a group of genes, probably a short section of chromatin usually inherited

intact, which bears Tripsacum genes, or genes with Tripsacum effects. We assume that these segments of chromatin have been received originally from Tripsacum as the result of translocations between Zea and Tripsacum chromosomes.

Since these translocation segments are usually inherited intact, they may be treated as single genes. Linkage studies with the genes which had previously been used to mark the four chromosomes included in these studies show that none of the segments occur on chromosomes 2, 6 or 9, but that one, possibly two, of them are located on chromosome 4. Translocation segment A shows 29 per cent of crossing-over with *Tu* and 34 per cent with *su*. Segment C shows 38 per cent of crossing-over with *su* and 47 per cent with *Tu*. T-A and T-C exhibit 53 per cent of crossing-over with each other and *su* and *Tu* show 28 per cent.

Though the classification of this population into 16 distinct classes was achieved only by concentrating on the essential differences between Zea and Euchlaena and ignoring all others, yet once the classification was made, it was found that all other differences previously measured were associated with these four translocation segments. There is, in most cases, a linear relationship between the number of translocation segments and the quantitative characteristics which differentiate the two genera. In several characteristics the translocation segments account for almost all of the differences between Zea and Euchlaena. We may conclude, therefore, that Euchlaena differs from Zea primarily by four segments of chromatin, which, because they bear genes with Tripsacum effects, are assumed to have been received originally from Tripsacum as the result of natural hybridization of Zea and Tripsacum followed by back-crossing to Zea. Two of these segments, which account for approximately half of the differences between Zea and Euchlaena, appear to be located at opposite ends of Chromosome No. 4 and this accounts for the fact, previously noted, that the genes which differentiate Zea and Euchlaena are not distributed at random but are concentrated on chromosome No. 4.

Though the archaeological problems of Central America are far from final solution, there are several well-established facts, which, considered with the botanical evidence, suggest that the hybridization of Zea and Tripsacum which give rise to the new genus Euchlaena probably occurred after the year 600 A. D. Euchlaena is unknown and Tripsacum is not abundant in the regions of the Old Maya Empire in Guatemala and the New Empire in Yucatan. When the Mayas abandoned the Old Empire cities about 600 A. D. (cf. Spinden,[16] Morley[17]) they moved in two directions, north to Yucatan and west to the highlands. At least one city in western Guatemala, Quen Santos (cf. Seler[18]), is characterized by the same dated monuments which appeared in all the larger Maya cities and the earliest date inscribed at Quen Santos is identical with the date on the lintel at Chichen

Itza, and both are exactly 10 tuns (3600 days) later than the last dates at the two Old Empire cities, Seibal and Tikal. The only place in America where Euchlaena has been found growing as the dominant species is in the highlands of western Guatemala not far from the ruins of Quen Santos. Tripsacum grows in profusion here; and here, too, is the only region where the natives have a Maya word, *salic* or *salicim*, for Euchlaena.[6] All of these facts suggest, though do not prove, that Euchlaena originated after the Mayas abandoned the Old Empire and migrated to the highlands bringing Zea and Tripsacum into direct contact with each other on a large scale.

With Euchlaena eliminated as a recent development, there remains no reason for dismissing pod-corn as the putative ancestor of cultivated maize. In the homozygous state pod-corn is often an earless, perfect-flowered plant bearing its seeds in the terminal inflorescence or tassel.[10] It possesses the two essential characteristics which maize now lacks and which it must have had in order to survive in the wild, a protection for the individual seed and a means of dispersal. The first is provided by prominent glumes, the second by a brittle rachis. Since we have at hand in pod-corn all of the characteristics which the morphologists demand in the progenitor of maize, and since in its homozygous form pod-maize is identical with other cereals, in its brittle rachis, perfect flowers and glume-enclosed seeds, and finally since the very existence of this peculiar form requires an explanation, we see no necessity for searching further for the prototype of maize.

The change from pod-corn to naked corn is the result of a recessive single gene mutation, a mutation which may have occurred repeatedly in the past only to be lost in nature. Once the mutation occurred under domestication, however, it immediately acquired a survival value which it had never previously possessed. Many other changes must have followed this original mutation, one of these being the shortening of the axes of the lateral branches so that the lateral inflorescences are now enclosed in the overlapping leaf sheaths, or shucks. The pod-corn of today, therefore, is the result of superimposing upon a plant, which has been tremendously altered by millennia of domestication, a single relic wild gene, and though modern pod-corn has most of the essential characteristics of the original wild corn, it must differ greatly from the wild plant in many others.

There is also no longer any necessity for seeking the center of origin of maize in Mexico or Central America; most of the evidence, except its close relationship to teosinte, has consistently pointed to Peru as the primary center of domestication. Here appears the first serious discrepancy in the argument, for Peru is one of the few places in America where pod-corn is unknown. This may be accounted for by assuming that pod-corn has survived as a relic gene in the maize varieties of the more primitive Indians, who, notoriously careless in maintaining purity in their varieties, have

never practiced rigid selection against the podded characteristic, but that in Peru where agriculture reached a higher stage of advancement than elsewhere in America, selection finally succeeded in eliminating the podded gene from the population. If this assumption is sound, we might still expect to find pod-corn represented in the prehistoric Peruvian pottery, for the ancient Peruvians depicted many of their food plants on their pottery and some of the representations, particularly those found on the Chimu pottery of the coast, are extremely realistic. Accordingly we began a search in the museums, and among the illustrations in the archaeological literature for representations of pod-corn, a search which reached fruition in the discovery at the Peabody Museum at Yale University of a faithful replica of a prehistoric ear of pod-corn.

Although Peru was undoubtedly the primary center of domestication of maize, it probably was not the original wild habitat, for the geography and climate of Peru are not those under which we should expect maize to have existed as a wild plant. It appears far more likely that maize had its native habitat in the lowlands of South America, not in the tropical rain forests, for maize competes but poorly with tall vegetation and its seed have no mechanism for maintaining dormancy under humid conditions. But the forests of South America are interspersed with rainy-green savannas where an abundance of rain is followed by a dry period.[19] In such regions a wild maize plant might easily persist. Supporting this assumption are found four independent historical references associating pod-corn with the Guarany Indians of South America or with the region which they occupied.[7,20,21,22] One of these,[21] describes a type of maize which bears its seeds in the tassel and which may well have been the homozygous, earless form of pod-corn.

There is no evidence in conflict with the view that maize had but one center of origin. In the case of several other crops, notably beans and squashes, it appears that domestication proceeded independently in Peru and Central America, but this is not true of maize. There is, however, rather conclusive evidence that hybridization of Zea and Tripsacum which produced Euchlaena, and subsequent repeated hybridizations of Euchlaena with its Zea parent, have given rise to new types of maize previously not in existence. We should have expected this on *a priori* grounds and on the same grounds we should conclude that the pointed-seeded pop corns represent one of the products of this hybridization. Kuleshov's[23] studies of the Russian world collection of corn varieties show that the pointed pop corns are almost unknown in Guatemala and Peru, but occur in greater diversity in Mexico than in all of the rest of the world combined. The long slender straight-rowed flint and flour corns, which are also putative products of Tripsacum contamination, are also rare in Guatemala or Peru, and in this case also in Mexico, but make their first appearance in the Pueblo region

of the Southwest at about the year 900 A. D. It appears to us that almost all of the North American maize varieties exhibit a Tripsacum influence and that the chief differences between North and South American varieties are attributable to an infection of Tripsacum genes in the former; its absence in the latter.

This hypothesis suggests a number of genetic and cytological tests, one of the most obvious being a comparison of the knobs on the chromosome of North American and South American varieties. The presence of chromosome knobs is one of the many characteristics in which Euchlaena is intermediate between Zea and Tripsacum.[24] If the knobs on the chromosomes of Euchlaena were derived originally from Tripsacum, some species of which possess knobs on every chromosome, and if knobs on the chromosomes of North American maize varieties were also received originally from Tripsacum by way of Euchlaena, we might expect the indigenous South American maize varieties to be completely lacking in knobs.

At this writing we have examined the prophase chromosomes of only one South American variety, the well-known giant-seeded Cuzco, a variety which is definitely of South American origin. The chromosomes of Cuzco were found to be completely lacking in knobs. If this proves to be a constant characteristic of South American varieties, it will serve as a useful criterion in distinguishing pure maize from Tripsacum-infected varieties and will aid in tracing the movements of maize in America in prehistoric times.

The tripartite hypothesis that cultivated maize had its origin in South America as a single gene mutation from a wild form of pod-corn; that Euchlaena is a recent product of the natural hybridization of Zea and Tripsacum which occurred when the two genera were brought together in Central America, and that new types of maize originating from this cross comprise the majority of North American varieties, is in accord, we believe, with all the known facts. Its value as a working hypothesis has already been established by the discovery of pod-corn in the prehistoric Peruvian pottery and of maize from South America with knobless chromosomes, discoveries which had been predicted on hypothetical grounds. The chief additional evidence which might be demanded, the actual collection of pod-corn in the wild in South America, may also be forthcoming when we begin to seek this plant in the unforested lowlands of Southwestern Brazil, Bolivia or Paraguay.

[1] DeCandolle, *Géographie Botanique*, Paris (1855).
[2] Darwin, *The Variation of Animals and Plants under Domestication*, London (1868).
[3] Laufer, *Congrés Internat. d. Américanists*, **15** [1] 223–257 (1907).
[4] Knowlton, *Jour. Wash. Acad. Sci.*, **9**, 134–136 (1919).
[5] Brown, *Ibid.*, **24**, 293–296 (1934).
[6] Kempton and Popenoe, *Carnegie Inst. of Wash. Publ.*, **483**, 199–218 (1937).

[7] Saint-Hilaire, *Ann. d. Sci. Nat.*, **16**, 143–145 (1829).
[8] Sturtevant, *Bull. Torrey Bot. Club*, **21**, 319–343 (1894).
[9] Harshberger, *Garden and Forest*, **9**, 522–523 (1896).
[10] Collins, *Jour. Wash. Acad. Sci.*, **21**, 520–530 (1912).
[11] Weatherwax, *Amer. Midland Naturalist*, **16**, 1–71 (1935).
[12] Spinden, *Proc. 19th Int. Cong. Americanists*, 269–276 (1917).
[13] Emerson and Beadle, *Zeitschr. f. indukt. Abstamm. u. Vereb.*, **62**, 305–315 (1932).
[14] Beadle, *Genetics*, **17**, 481–501 (1932).
[15] Mangelsdorf and Reeves, *Jour. Hered.*, **22**, 329–343 (1931).
[16] Spinden, *A Study of Maya Art*, Cambridge, Mass. (1913).
[17] Morley, *Carnegie Inst. of Wash. Publ.*, **219**, 1–643 (1920).
[18] Seler, *Die alten Ansiedelungen von Chacula*, Berlin (1901).
[19] Weberbauer, *Field Mus. Nat. Hist. Publ.*, **351**, 13–81 (1936).
[20] Dobrizhoffer, *An Account of the Abipones*, London (1822).
[21] Azara, *Voyages dans l'Amérique Méridionale*, Vol. I, Paris (1809).
[22] Bonafous, *Histoire naturelle, agricole et economique du Mais*, Paris (1836).
[23] Kuleshov, *Jour. Am. Soc. Agron.*, **25**, 688–700 (1933).
[24] Longley, *Jour. Ag. Res.*, **54**, 835–862 (1937).

CALCULATIONS OF BIOELECTRIC POTENTIALS. III. VARIATION IN PARTITION COEFFICIENTS AND ION MOBILITIES

BY S. E. HILL AND W. J. V. OSTERHOUT

LABORATORIES OF THE ROCKEFELLER INSTITUTE FOR MEDICAL RESEARCH

Communicated June 24, 1938

Very interesting differences are found between the cells of *Nitella flexilis*, Ag., in current use and those employed in earlier experiments.[1] The earlier cells came from a different locality and for convenience we shall call them Lot A and the later ones Lot B.

In the earlier cells Na^+ and K^+ seemed to have very different mobilities in X, the outer non-aqueous protoplasmic surface layer. But in the later cells these mobilities, U_K and U_{Na}, are not far apart. We therefore expected much less change in P.D. on replacing NaCl by KCl but were surprised to find that the change is somewhat greater.

This can be accounted for if we assume that in the later cells the partition coefficient[2] S (concentration in X ÷ concentration in the external solution) is much greater for KCl than for NaCl.

In the previous studies[1] the P.D. could be satisfactorily predicted by regarding it as due to diffusion potential and hence obeying the equations of Nernst and of Henderson.

The apparent mobility was calculated from the equation[3] of Nernst which may be written (for 20°C.)

$$\text{P.D.} = 58\frac{u - v}{u + v}\log\frac{C_1}{C_2}$$

where C_1 and C_2 are concentrations, u is the mobility of the cation and v that of the anion: for convenience we put $v = 1$ and calculate the value of u.[4]

In Lot A it was found[1] that the replacement of 0.001 M by 0.01 M gave a change of P.D. of 20.9 mv. for NaCl and of 54.7 mv. for KCl. From this we get $u_{Na} = 2.18$ and $u_K = 73.24$. In Lot B[5] the corresponding values[6] are 45 and 49, giving $u_{Na} = 7.93$ and $u_K = 11.9$.

Using these values we may calculate the changes of P.D. to be expected when 0.01 M NaCl is replaced by 0.01 M KCl (this is called the potassium effect). For this purpose we make use of Henderson's equation. This may be written (for 20°C.)

$$\text{P.D} = 58\frac{(U_I - V_I) - (U_{II} - V_{II})}{(U_I + V_I) - (U_{II} + V_{II})}\log\frac{U_I + V_I}{U_{II} + V_{II}}$$

where $U_I = u_K C_K$, $U_{II} = u_{Na} C_{Na}$, $V_I = v_{Cl} C_K$ and $V_{II} = v_{Cl} C_{Na}$: all values apply to the protoplasmic surface, X; u_K and u_{Na} denote mobilities and C denotes concentration.

In dealing with Lot A it was assumed that the partition coefficient S was the same for KCl and NaCl so that in X we should have $C_K \div C_{Na} = 1$. It was then found that the calculated value was not far from the value of 83 mv. observed[7] on substituting 0.01 M KCl for 0.01 M NaCl.

In Lot B the calculated value on this basis is 9.3 mv. and the observed value 95 ± 2 mv. (120 observations on 44 cells) but if we suppose that S is greater for KCl than for NaCl so that in X we have $C_K \div C_{Na} = 60$ the calculated value[8] becomes 95 mv.

It would not be surprising if the partition coefficient for potassium were higher than for sodium since potassium is usually taken up in preference to sodium by living cells. It must be remembered, however, that we are here dealing with the concentration of K^+ and Na^+ in X rather than with that of KCl and NaCl: the latter may be weak electrolytes in X and may have different dissociation constants.[9]

These calculations indicate that the partition coefficient can vary greatly and this seems highly probable in view of experiments with models which show that the partition coefficients of organic and inorganic substances between aqueous and non-aqueous phases can be greatly influenced by the addition of small amounts of reagents.[10] If the protoplasmic surface layer is non-aqueous, as we suppose, it is easy to see how great changes in partition coefficients might occur as the result of metabolism.

Not only the partition coefficients but also the mobilities differ from those found in Lot A. In Lot A we have $u_K = 73.24$, $u_{Na} = 2.18$; in Lot B,

$u_K = 11.9$ and $u_{Na} = 7.93$. Such differences[11] are not surprising in view of the fact that the removal of organic substances from the cell by distilled water can reduce[12] u_K from 85 to 2 or less: this can be restored by a variety of substances.[13] Changes can also be produced by guaiacol.[14] Hence it seems possible that alterations in metabolism can affect mobilities as well as partition coefficients.

Summary.—From changes in P.D. produced by changes of concentration we can calculate the apparent mobilities of K^+ and Na^+ in the outer protoplasmic surface of *Nitella*. We can thus predict the potassium effect (the change in P.D. produced by substituting 0.01 *M* KCl for 0.01 *M* NaCl) by assuming a value for the ratio of partition coefficients $S_{KCl} \div S_{NaCl}$ (S = concentration in the outer non-aqueous protoplasmic surface layer ÷ concentration in the external solution).

In cells studied formerly it was assumed that $S_{KCl} \div S_{NaCl} = 1$ but in the cells used in the present investigation we assume that $S_{KCl} \div S_{NaCl} = 60$.

Such variations in S do not appear improbable in view of experiments on models.

In the cells studied earlier the apparent mobilities of K^+ and Na^+ differed from those found in the present investigation. This is not surprising as alterations of mobilities can be brought about by a variety of reagents and the apparent mobilities might therefore be expected to vary according to the metabolism of the cell.

[1] Cf. Osterhout, W. J. V., *Jour. Gen. Physiol.*, **13**, 715 (1929–1930).

[2] The term partition coefficient is used in liberal sense. If we have KCl in the external solution K^+ in X may be to some extent paired with organic anions or may form complexes in the sense of Kraus and Fuoss (Kraus, C. A., *Trans. Electrochem. Soc.*, **66**, 179 (1934); Fuoss, R. M., *Chem. Rev.*, **17**, 27 (1935).

[3] It is assumed that the partition coefficient, S, is constant which is probably not far from true in this case. In models S is relatively constant for NaCl and KCl as compared with sodium guaiacolate and potassium guaiacolate. Concentrations are employed for convenience in place of activities.

[4] All values relate to X.

[5] For the technique employed in Lot B, see Hill, S. E., and Osterhout, W. J. V., *Jour. Gen. Physiol.*, **21**, 541 (1937–1938). All measurements were made from photographic records.

[6] For NaCl 45 ± 0.69 mv. (75 observations on 37 cells) and for KCl 49 ± 0.49 mv. (95 observations on 41 cells).

[7] In all cases KCl is negative to NaCl in the external circuit.

[8] For convenience we put $C_{Na} = 1$, $V_{Cl} = 1$. We then have $U_I = 11.9\ (60) = 714$, $V_I = (1)\ 60 = 60$, $U_{II} = 7.93\ (1) = 7.93$ and $V_{II} = 1\ (1) = 1$.

[9] We suppose that X is a non-aqueous layer with a low dissociation constant (Osterhout, W. J. V., *Ergebn. Physiol.*, **35**, 967 (1933); *Trans. Faraday Soc.*, **33**, 997 (1937); Shedlovsky, T., and Uhlig, H. H., *Jour. Gen. Physiol.*, **17**, 549, 563 (1933–1934)).

[10] Unpublished. See also Hill, S. E., and Osterhout, W. J. V., *Jour. Gen. Physiol.*, **21**, 553 (1937–1938) (footnote 29).

[11] These mobilities when determined from concentration effects are largely independent of partition coefficients. See footnote 3.

[12] Osterhout, W. J. V., *Jour. Gen. Physiol.*, **18**, 992 (1934–1935).

[13] Cf. Osterhout, W. J. V., and Hill, S. E., *Proc. Soc. Exptl. Biol. and Med.*, **32**, 715 (1934–1935).

[14] Unpublished. For changes in mobilities produced by guaiacol in *Valonia*, see Osterhout, W. J. V., *Jour. Gen Physiol.*, **20**, 13, 685 (1936–1937).

RAYLEIGH WAVES

BY H. BATEMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated June 18, 1938

1. *The Interaction of the Air and the Ground.*—The transmission of plane longitudinal waves of unlimited extent from the ground to the air was investigated by C. G. Knott[1] many years ago. He found that the resulting air waves, which are propagated in almost a vertical direction, generally have only a small amount of energy. Earthquake sounds have been studied by many writers. C. Davison[2] put forward the theory that they originate from the margin of the region disturbed by the earthquake and travel some distance through the earth before being transmitted to the air. A summary of results relating to earthquake noises has been given by Landsberg.[3] The type of air motion considered here is not the simple progressive wave in an unlimited atmosphere but is a type of free vibration of the air and ground having the characteristics of a Rayleigh wave except that its velocity of propagation is less than the velocity of sound in air instead of being slightly less than the velocity of a shear wave in the ground. The mathematical analysis is very similar to that used by Stoneley[4] in his study of Rayleigh waves in a plane homogeneous elastic earth below a compressible sheet of water of unlimited extent. It is assumed here, however, that the vertical velocity of the air is negligible at a height H above the ground while in Stoneley's work the boundary condition at the free surface of the water is one of constant pressure. His remarks on nodal planes indicate that his analysis may be applicable in our case but it has been thought worth while to give the analysis again in a form in which the velocity of the wind is taken into consideration and some of Stoneley's approximations are omitted. It is thought that the analysis may be of some interest in connection with the interpretation of the ground roll observed in geophysical field work. For information relating to the ground roll I am indebted to Dr. Gutenberg, Mr. Martin Gould and other members of the group connected with the Pasadena Seismological Laboratory. It

has generally been assumed, of course, that the influence of the air on the propagation of seismic waves is slight but such an assumption ought to be justified by numerical work in the case of waves produced by an artificial explosion for there are some features of the phenomena that are not fully elucidated. The problem resembles that of the loud speaker with infinite baffle, the disturbed area of the earth corresponding to the membrane that is set in vibration. Now in the theory of the loud speaker the short circuiting of energy is a familiar phenomenon, there is not simply a radiation of energy outwards. If, then, there is a similar short circuiting of energy in the air after an explosion, an interaction of air and ground is to be expected. If the air and ground are treated as a coupled system, an explosion may be expected to give rise to a subsequent motion that is composed of free vibrations of the system and the particular type of motion to be studied is, indeed, a free vibration. The whole problem is, then, one of the partition of energy among a number of free vibrations including in particular the ordinary Rayleigh wave and the new type of Rayleigh wave. This second type of Rayleigh wave is called "new" merely to distinguish it from the old type but it has been known for a long time that there is more than one type of Rayleigh wave for a stratified medium. With regard to the likelihood of the existence of a marked interaction between the earth and the air it should be mentioned that many years ago the late Lord Rayleigh[5] concluded that in the vicinity of a vibrating body of linear dimensions small in comparison with the wave-length, the air acts as if it were almost incompressible while the great mass of air at some distance from the body is slightly compressed periodically. A similar conclusion has been reached more recently by Lennard Jones[6] after some elaborate calculations. In trying to apply this result to our problem we are led to surmise that when the ground rises initially after an explosion the air immediately above it will either move away laterally and produce a reaction on the ground somewhere else or will try to lift or compress the great body of air above it.

2. *The Problem of a Flat Earth and a Homogeneous Atmosphere.*—Taking the axis of x along the ground in the direction of propagation and the axis of z vertically downward, the component displacements, u, w, in the earth, which for simplicity is supposed to be homogeneous, can, as in the theory of the late Lord Rayleigh,[7] be expressed in the form

$$u = u_1 + u_2, \qquad w = w_1 + w_2,$$

where

$$u_1 = ifPe^{-rz - ifX}, \; w_1 = rPe^{-rz - ifX}$$
$$u_2 = isQe^{-sz - ifX}, \; w_2 = fQe^{-sz - ifX}$$
$$X = x - vt, \; v = \text{velocity of propagation},$$
$$r^2 = f^2(1 - v^2/a^2) = f^2C^2(v), \text{ say},$$
$$s^2 = f(1 - v^2/b^2) = f^2S^2(v), \text{ say}.$$

The symbol a is used here to denote the velocity of propagation of longitudinal waves and the symbol b to denote the velocity of propagation of waves of shear.

Supposing for simplicity that the undisturbed air has a velocity V (independent of z, x and t) parallel to the axis of X, a constant density ρ and a constant velocity of sound c, the velocity potential, ϕ, of a small irrotational perturbation satisfies the partial differential equation

$$c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2}\right) = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2\phi.$$

This equation has a solution of the form

$$\phi = Ae^{-mz - ifX} + Be^{mz - ifX}$$

in which A, B, m are constants, if m is given by the equation

$$m^2 = f^2[1 - V^2/c^2] = f^2M^2(v), \text{ say.}$$

When $(v - V)^2 < c^2$, m is real and the air wave may be regarded as of Rayleigh's type. When $(v - V)^2 > c^2$, the air wave consists of a wave of sound and its reflection at some reflecting layer.

The component velocities of the air are U, W, where

$$U = \frac{\partial\phi}{\partial x} = -if(Ae^{-mz} + Be^{mz})e^{-ifX}$$

$$W = \frac{\partial\phi}{\partial z} = m(Be^{mz} - Ae^{-mz})e^{-ifX}.$$

The condition $W = 0$ when $z = -H$ gives $A = Ke^{-mH}$, $B = Ke^{m}$, where K is a quantity which may be found from the condition that

$$W = \frac{\partial w}{\partial t} \text{ when } z = 0.$$

Thus $$2mK\,sh(mH) = ifv(rp + fQ).$$

When the viscous drag of the air on the ground is neglected the absence of a shearing traction on the ground gives the equation

$$2frP + (s^2 + f^2)Q = 0.$$

Since the normal excess pressure of the air on the ground due to the perturbed air motion is wholly responsible for the normal traction on the ground, we find that

$$\sigma a^2\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) - 2\sigma b^2\frac{\partial u}{\partial x} = \rho\left(\frac{\partial\phi}{\partial t} + V\frac{\partial\phi}{\partial x}\right)$$

where σ is the density of the ground. Hence

$$\sigma a^2(f^2 - r^2)P - 2\sigma b^2 f(fP + sQ) = 2Kif\,\rho(v - V)\ \text{ch}\ (mH).$$

Substituting the values already found for the ratios of P, Q, K, we obtain the equation

$$\sigma b^4 M(v)\ \tanh\ [fHM(v)].R(v) = \rho v^3(v - V)\ C\ (v) \qquad \text{(A)}$$

where $$R(v) = 4S(v)C(v) - [1 + S^2(v)]^2.$$

This equation may be discussed in much the same way as Stoneley's equation. For our present purpose we remark first that when $v < c + V$ and both quantities are less than the velocity v_1 of the ordinary Rayleigh wave which makes $R(v_1) = 0$, then $R(v) > 0$ and if also $v > V$ both sides of equation (A) are positive. If, moreover, v is chosen so that

$$\sigma b^4 M(v)R(v) > \rho v^3(v - V)C(v)$$

there is a value of $\tanh\ [fHM(v)]$ equal to a positive proper fraction for which equation (A) is satisfied and so H can be found uniquely when v is given. The critical value of v is that given by the equation

$$\sigma b^4 M(v)R(v) = \rho v^3(v - V)C(v).$$

With ordinary values for the elastic constants of the ground this critical value v_c is only slightly less than $c + V$. When $v < v_c$ there is a Rayleigh wave in the ground as well as in the air.

Since r and s decrease as v increases from 0 to b, the amplitudes of the constituents of a Rayleigh wave in the ground drop more rapidly with increasing depth when v is close to c than when v is close to v_1. The drop in the amplitude of the Rayleigh wave in the air is not quite exponential for

$$U = -2ifK\ \text{ch}\ (mz + mH)\ e^{-ifX},\ W = 2mK\ \text{sh}\ (mz + mH)e^{-ifX}.$$

When $c + V < v < v_1$ and $N^2(v) = (v - V)^2/c^2 - 1$, the equation for H may be written in the form

$$\sigma b^4 N(v)\ \tan\ [fHN(v)]R(v) = -\rho v^3(v - V)C(v). \qquad \text{(B)}$$

When v is assigned $\tan\ [fHN(v)]$ has a definite sign when the foregoing equation is satisfied and so the height H of the reflecting layer is confined to certain ranges that are equally spaced. This may mean that a value of v for which $c + V < v < v_1$ can only exist under certain atmospheric conditions. To find these it will be necessary to assume that the temperature and wind velocity vary with altitude. The equation for ϕ is then much harder to solve.

The existence of a large rock or mountain from which air waves can be reflected may also be influential in determining the possible values of v but then the problem of propagation must be studied in three dimensions.

3. *Remarks on the Theory of the Ground Roll.*—Though some seismolo-

gists think that the ground roll has some of the characteristics of a Rayleigh wave yet another explanation has been sought on account of the low velocity of the ground roll as compared with that of the ordinary Rayleigh wave. The existence of a second type of Rayleigh wave which arises from a marked interaction between the air and the earth puts a different complexion on the matter. The part which this type of free vibration plays can only be judged by a study of the waves produced by a shock of short duration. An extension of the work of Lamb,[8] Banerji,[9] Coulomb[10] and the Japanese seismologists[11] is necessary. The work of Jeans,[12] in which seismic waves are regarded as free vibrations (of high order) of a spherical earth must also be extended. Such extensions may be useful for a study of Stoneley's problem and for a study of the effect of a swiftly moving river on the propagation of Rayleigh waves.

The results of Lamb's calculations are only partly supported by the observations.[13] Andreotti,[14] who has used them in his study of the microseisms produced by waves of the Adriatic dashing against the coast, thinks that there is good agreement. His numerical calculations are based, however, on the assumption that the whole coast is affected by a wave in one phase all along its length. When, moreover, waves dash against a cliff, diffraction effects must be taken into consideration; the problem is not simply one of a solid with plane face. It is possible, however, that edge effects can be taken into consideration in the simple problem by using surface singularities represented by wave-potentials of type

$$(x + iz)^{-1/2}F(t - \omega/a),$$

where $\omega^2 = x^2 + z^2$. In this connection it is interesting to note that a relativity transformation gives a corresponding solution for a similar type of source moving with velocity v and an integration over a set of such moving sources, with an appropriate choice of the individual functions F, leads to the wave-function

$$\int_{-\infty}^{\infty} [z + 1s - i(x - vt)(1 - v^2/a^2)]^{-1/2} e^{-iks} ds,$$

which, when evaluated, leads to just the type of wave-function used in the representation of Rayleigh waves in two dimensions.

The study of elastic waves in two dimensions is complicated by the lack of simplicity in the solution for a line source. Various forms of the principle of Huygens have been found by Volterra,[15] Picht[16] and others, but have not been used very often. Picht[17] has, however, tried to use the principle of Huygens to interpret theoretically the boundary wave which in a stratified earth, travels with the velocity of the upper medium. It is assumed that the strongly damped wave issuing from the focus sets the boundary between the two media into vibration. Such an assumption is very similar to our

assumption that an explosion in the earth sets the boundary between air and earth into vibration producing a wave which travels with a velocity close to that of the upper medium (air). Dr. Gutenberg has furnished me with the information that the velocity of the ground roll is sometimes greater than the velocity of sound in air but is usually between 100 meters a second and 500 meters a second. The amplitude of the ground roll decreases when the depth of the source increases, being small when the depth is 10 meters and negligible when the depth is 20 meters or more. Banerji's calculations indicate a marked decrease in amplitude of the ordinary Rayleigh wave when the depth of the source increases and we might expect by analogy that there would be a similar decrease of amplitude for the Rayleigh wave of low speed. An attempt must therefore be made to test this surmise. Banerji's calculations were for the three dimensional case to which we must now turn. Here there is the advantage that there is a simple wave function for a point source but much analytical ground work is needed before a complete solution of a problem can be obtained. Coulomb[10] has made some progress by endeavoring to extend Boussinesq's solution of statical elastic problems to wave problems. Much of the analysis depends on the properties of a certain transcendental function which he has studied in some detail. Some further properties of this function are developed in an accompanying paper.

[1] C. G. Knott, *Phil. Mag.* (5) **48,** 64–97 (1899).

[2] C. Davison, *Ibid.* (5) **49,** 31–70 (1900).

[3] H. Landsberg, *Trans. American Geophysical Union,* Part 1, 118–120 (1937).

[4] R. Stoneley, *Monthly Notices Royal Astronomical Soc., Geophys. Suppl.,* **1,** 349–356 (1926).

[5] Lord Rayleigh, *Theory of Sound,* Vol. 2, p. 158; Sc. Papers, Vol. 4, No. 230.

[6] J. E. Lennard Jones, *Proc. London Math. Soc.* (2) **20,** 347–364 (1920).

[7] Lord Rayleigh, *Ibid.* (1) **17,** 4–11 (1885); Sc. Papers, Vol. 2, p. 441.

[8] H. Lamb, *Phil. Trans. Roy. Soc. London* (A) **203,** 1–42 (1904).

[9] S. K. Banerji, *Phil Mag.* (6) **49,** 65–80 (1925).

[10] J. Coulomb, *Am. de Toulouse* (3) **23,** 91–137 (1931).

[11] H. Nakano, *Japanese Jour. of Astronomy and Geophysics,* **2,** 233–326 (1925); *Geophysical Mag.,* **1,** 255–303 (1926–1928); **2,** 189–348 (1930); H. Arakawa, *Geophysical Mag.,* **7,** 155–160 (1933); K. Sezawa and G. Nishimura, *Bull. Earthquake Research Institute Tokyo Imp. Univ.,* **7,** 41–64 (1929).

[12] J. H. Jeans, *Proc. Roy. Soc. London* (A) **102,** 554–574 (1923).

[13] J. B. Macelwane, *Bull. National Research Council,* No. 90, 121–128 (1933).

[14] G. Andreotti, *Veneto Atti,* $\mathbf{91_2}$, 1345–1358 (1932).

[15] V. Volterra, *Acta Math.,* **18,** 161–232 (1894).

[16] J. Picht, *Zeit. f. Physik,* **91,** 717–723 (1934).

[17] J. Picht, *Ann. d. Physik* (5) **19,** 913–920 (1934); *Beiträge Angew. Geophysik,* **3,** 1–8 (1933).

COULOMB'S FUNCTION

BY H. BATEMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated June 18, 1938

1. In his work on Rayleigh waves Coulomb[1] has studied the function

$$\psi_n(hw,\ hw \operatorname{sh} a) = i^n \int_a^\infty e^{-ihw \operatorname{ch} u} \operatorname{ch}(n\,u)du. \qquad (1.1)$$

$$\operatorname{ch} a = \cosh a,\ \operatorname{sh} a = \sinh a$$

The function ψ_0 with a complex value of h occurs in the work of Buchholz[2] on the propagation of alternating currents in the earth between two electrodes connected above ground by a rectangular loop of wire whose vertical ends support the horizontal piece. Use will be made here of the notation

$$C_n(a, x) = \int_a^\infty e^{-x \operatorname{ch} u} \operatorname{ch}(n\,u)du,\ S_n(a, x) = \int_a^\infty e^{-x \operatorname{sh} u} \operatorname{ch}(nu)du \qquad (1.2)$$

wherein $R(x)>0$ and $a>0$. When $n = 0$ expansions of these functions are readily obtained by putting $x \operatorname{ch} u = v$, $x \operatorname{ch} a = c$ in the first integral and $x \operatorname{sh} u = w$, $x \operatorname{sh} a = s$ in the second. With the notation $(m/, n)$ for the binomial coefficient $C_{m,n}$ and the notation

$$Q(z, k) = \int_z^\infty e^{-t} t^{k-1} dt \qquad (1.3)$$

for the incomplete Gamma function of the second kind, the expansions obtained by using the binominal theorem are

$$C_o(a, x) = \sum_{n=0}^{\infty} (-)^n (-{}^1/_2/, n) x^{2n}\, Q(c, -2n) \qquad (1.4)$$

$$S_o(a, x) = \sum_{n=0}^{\infty} (-{}^1/_2/, n) x^{2n}\, Q(s, -2n).$$

The first of these expansions is given by Coulomb and Buchholz. The convergence of the series may be established by using the formula

$$Q(c, -m) = e^{-c} c^{-m}/m - e^{-c} c^{-m+1}/m(m-1) + \ldots (-)^m Q(c, o)/m!$$

The second series converges absolutely when $\operatorname{sh} a>1$. When each term is transformed by using the formula

$$\Gamma(2n+1) e^s\, Q(s, -2n) = \int_0^\infty e^{-st}\, t^{2n}\, dt/(1+t)$$

as in Buchholz's transformation of the series for $C_o(a, x)$, we find that

$$S_o(a,x) = \int_0^\infty \exp\ [-(1+t)x \text{ sh } a]\ J_o\ (xt)\ dt/(1+t), \quad (1.5)$$

while the corresponding formula of Buchholz is

$$C_o\ (a,\ x) = \int_0^\infty \exp\ [-\ (1+t)x \text{ ch } a]\ I_o\ (xt)\ dt/(1+t). \quad (1.6)$$

It should be noticed that by expanding $1/(1+t)$ in powers of t and integrating term by term we obtain the same asymptotic series for $S_o(a,x)$ as is obtained from (1.2) by repeated integration by parts.

A relation between $C_n(a,x)$ and $S_n(a,x)$ may be found by putting $s = \text{sh } u$ in the integral

$$\int_0^\infty e^{-sz}\ J_o[(z^2+2xz)^{1/2}]\,dz = (1+s^2)^{-1/2} \exp[-x\{(1+s^2)^{1/2}-s\}], \quad (1.7)$$

multiplying by ch $nu\ e^{-x \text{ sh } u}\ du$ and integrating from a to ∞. This gives

$$C_n(a,x) = -(d/dx)\int_a^\infty e^{-x \text{ ch } u}\text{ch}(nu)du/\text{ch } u \quad (1.8)$$

$$= -(d/dx)\int_0^\infty S_n(a, x \text{ sh } a \text{ ch } v)J_o\ (x \text{ sh } v)x \text{ sh } v\ dv.$$

If, on the other hand, we multiply (1.7) by $e^{-x \text{ sh } u}$ ch $u\ du$ and integrate from a to ∞ we find that

$$C_o(a,\ x) = \int_0^\infty e^{-x \text{ ch } v \text{ sh } a} J_o(x \text{ sh } v)\ \tanh v\ dv. \quad (1.9)$$

2. Another expansion for $C_o(a,x)$ may be found by using the function

$$V_n(x) = \int_0^\infty e^{-xt}\ P_n\left(\frac{t-1}{t+1}\right) dt/(t+1) = e^x \int_x^\infty e^{-u} P_n(1 - 2x/u)du/u \quad (2.1)$$

which has been studied in a former paper.[3] If $V_n(x) = e^x W_n(x)$ there is an expansion

$$W_n(x) = \sum_{m=0}^{n} (-n/,\ m)(n+m/,\ m)x^m\ Q(x,\ -m) \quad (2.2)$$

a differential equation

$$x^2 W_n''' + (x^2+3x)W_n'' + (2x+1)W_n' - n(n+1)W_n = 0 \quad (2.3)$$

and recurrence relations

$$x(W_n' + W_{n-1}') = n(W_n - W_{n-1}).$$

$$(n+1)W_{n+1}' + nW_{n-1}' - (2n+1)W_n' = 2(2n+1)W_n,$$

$$(4n+2)x(W_n' + W_n) = (n+1)^2(W_{n+1} - W_n) + n^2(W_n - W_{n-1}),$$

$$x(W_{n+1}'' - W_{n-1}'') = 2(2n+1)W_n + W_{n-1}' - W_{n+1}',$$

$$n^2 W_{n-1} = 2x^2 W_n'' + [2x^2 - 2(n-1)x]W_n' - n(2x-n)W_n$$

$$(n+1)^2 W_{n+1} = 2x^2 W_n'' + [2x^2 + 2(n+2)x]W_n' + (n+1)(2x+n+1)W_n$$

$$(2n-1)(n+1)^2(W_{n+1} - W_n) - 2n(2n^2-1)(W_n - W_{n-1})$$

$$+ (n-1)^2(2n+1)(W_{n-1} - W_{n-2}) = 2x(4n^2-1)(W_n + W_{n-1}). \quad (2.4)$$

The differential equation for $W_n(x)$ is adjoint to the differential equation

$$x^2 Z_n''' + (3x - x^2)Z_n'' + (1-2x)Z_n' + n(n+1)Z_n = 0, \quad (2.5)$$

which is satisfied by the function $Z_n(x) = F(-n, n+1; 1,1; x)$ which was studied at the same time[3] as $V_n(x)$. This function $Z_n(x)$ may be used to obtain the representation

$$V_n(z) = \lim_{x \to 1} Z_n(-d/dx)\, V_0(zx). \quad (2.6)$$

The generating function of $W_n(x)$ suggests the expansion

$$\int_a^\infty e^{-z \operatorname{ch} u} du = (1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} W_n[z(\operatorname{ch} a - 1)] \quad (2.7)$$

which is certainly convergent when $z > 0$, $a > 0$ but may be valid under more general conditions.

3. In the physical investigations the wave potential connected with $C_o(a, x)\, C(a, x)$ is

$$W = \int_z^\infty e^{-kR} ds/R, \quad (3.1)$$

where k is a complex constant and $R^2 = s^2 + w^2 = s^2 + x^2 + y^2$, x, y and z being rectangular coördinates. With $s = w \operatorname{sh} u$, $z = w \operatorname{sh} a$ the integral is $C_o(a, kw)$ and if $r^2 = z^2 + w^2 = x^2 + y^2 + z^2 = w^2 \operatorname{ch}^2 a$ the expansion of the integral W is

$$W = e^{-kw}(1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} W_n(kr - kw)$$

$$= e^{-kw} \sum_{n=0}^{\infty} [(r-z)/w]^n [W_n(kr - kw) - W_{n-1}(kr - kw)] \quad z > 0 \quad (3.2)$$

where it is understood that $W_{-1}(x) \equiv 0$. It is thought that this expansion will converge rapidly. This surmise can be checked as soon as the tables of the function $W_n(x)$ have been completed.

If $v = w(\text{ch } a - 1) = r - w$, there is also an expansion

$$\int e^{-kw \text{ ch } u} du = \sum_{n=0}^{\infty} (-1/2/, n)(2w)^{2n} Q(kv, -n). \quad (3.3)$$

4. There is an integral relation

$$\int_0^{\infty} t^{s-1} W_n(t) dt = \Gamma(s) G_n(s) \quad R(s) > 0 \quad (4.1)$$

in which

$$s(1 + s)(2 + s) \ldots (n + s) G_n(s) = (1 - s)(2 - s) \ldots (n - s). \quad (4.2)$$

This is readily derived from (2.2) and suggests the new definition

$$W_n(t) = (1/2\pi i) \int_{c-i\infty}^{c+i\infty} t^{-s} \Gamma(s) G_n(s) ds \quad (4.3)$$

which, when $c > 0$, may be found directly by an attempt to solve the differential equation by means of a definite integral. The function $G_n(s)$ occurs as a coefficient in the expansion

$$(\text{ch } a - 1)^s \int_a^{\infty} (\text{ch } u - 1)^{-s} du = (1 - e^{-a}) \sum_{n=0}^{\infty} e^{-na} G_n(s). \quad 0 < s < 1 \quad (4.4)$$

5. Another representation of $W_n(x)$ which may be useful in finding new properties of the function is

$$W_n(x) = (\pi/x)^{1/2} \int_0^{\infty} J_o(u) I_{n+1/2}(u^2/8x) \exp(-u^2/8x) du. \quad (5.1)$$

This is valid so long as $R(x) > 0$. The formula may be checked by means of the recurrence formulae for $W_n(x)$ and the finite series for $I_{n+1/2}(z)$.

6. An asymptotic expansion for $V_n(x)$ for large values of z such that $R(z) > 0$ may be obtained from the series

$$V_n(z) = \sum_{m=0}^{n} (-n/, m)(n + m/, m) \int_0^{\infty} e^{-zt} (t + 1)^{-m-1} dt \quad (6.1)$$

by using the asymptotic expansion of each of the integrals, it is

$$V_n(z) \sim \sum_{r=0}^{\infty} \sum_{m=0}^{n} (-n/, m)(n + m/, m)(-m - 1/, r) z^{-r-1} r! =$$

$$\sum_{r=0}^{\infty} F_n(2r + 1) z^{-r-1} (-)^n \quad (6.2)$$

where $F_n(x)$ is the polynomial studied in a former paper.[4]

7. It follows at once from the formula

$$V_n(x) = \int_0^\infty e^{-t} Z_n(t) dt/(x + t) \tag{7.1}$$

that

$$V_n(x) = V_o(x) Z_n(-x) + p_{n-1}(x) + p_{n-2}(x) + \ldots + p_o(x), \tag{7.2}$$

where $p_{n-1}(x)$ is a polynomial of degree $n - 1$ in x. Hence as $x \to 0$ $V_n(x) - V_{n-1}(x) \to p_{n-1}(0)$. To verify that this is equal to $-2/n$ we may use the last recurrence formula (2.4), this value being readily found when $n = 1$ and $n = 2$. This leads to the formula

$$\lim_{x \to 0} [V_o(x) - V_n(x)] = 2\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right). \tag{7.3}$$

On account of this relation it is often convenient to transform a series of type $\sum c_n V_n(x)$ into one of type

$$\sum_{n=0}^{\infty} b_n [V_n(x) - V_{n-1}(x)]$$

on the understanding that $V_{-1}(x) = 0$. The convergence of the resulting series as $n \to 0$ may then be readily tested. In particular it is found that the series (3.2) fails to converge when $z = 0$ as is to be expected.

[1] J. Coulomb, *Annales de Toulouse* (3), **23,** 91–137 (1931).

[2] H. Buchholz, *Arkiv für Elektrotechnik*, **30,** 1–33 (1936).

[3] H. Bateman, *Duke Math. Jour.*, **2,** 569–577 (1936).

[4] H. Bateman, *Tôhoku Math. Jour.*, **37,** 23–38 (1933).

[5] S. O. Rice, *Bell System Technical Jour.*, **16,** 101–109 (1937).

[6] H. Bateman and S. O. Rice, *Amer. Jour. Math.*, **60,** 297–308 (1938).

[7] The function $C_o(a, x)$ occurs also in a paper by S. O. Rice[5] in which a transformation is given of van der Pol's expression for the value on the ground of the wave-function of a vertical dipole placed at the surface of a plane earth. Expansion for $C_o(a, x)$, $C_n(a, x)$ may also be found by using the integrals (14) and (15) in a recent paper by the author and S. O. Rice.[6]

FUNCTIONAL TOPOLOGY AND ABSTRACT VARIATIONAL THEORY

BY MARSTON MORSE

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Communicated June 28, 1938

This paper is concerned with a summary of results which will appear at length in a fasicule in the series "Mémorial des sciences mathématiques," Gauthier-Villars, Paris. Part of the theory is based on certain conceptions and theorems in group theory. These will be summarized first in order to make clear their independence from the rest of the theory.

1. *The Group Theory.*—Let G be an additive abelian operator group with coefficients δ in a field Δ. With certain of the elements u of G we associate an element $\rho(u)$ in a simply ordered set of elements $[\rho]$. The set $[\rho]$ may in particular be the set of real numbers. We term $\rho(u)$ the *rank* of u. The rank $\rho(0)$ shall not be defined. The elements of G with rank (with 0 added) will not in general form a group. A subgroup g of G will be termed an *operator* subgroup if when u is in g, δu is in g. The property A is termed an *operator property* if whenever u has the property A and $\delta \neq 0$, δu has the property A. By a subgroup g of G with property A is meant an operator subgroup every element of which with the possible exception of 0 has the property A. The group g will be termed *maximal* if it is a proper subgroup of no subgroup of G with property A. The ranks $\rho(u)$ shall satisfy the following three conditions:

I—If u has a rank and $\delta \neq 0$, $\rho(u) = \rho(\delta u)$

II—If u, v, and $u + v$ have ranks, then

$$\rho(u + v) \leqq \max [\rho(u), \rho(v)]$$

III—If u and v have unequal ranks $\rho(u + v)$ exists.

The proof of the following theorem is due to R. Baer.

THEOREM 1.1. *Let* g *be an operator subgroup of* G *whose dimension is at most alef-null. If each element of* g *save the null element has a rank,* g *is a direct sum of suitably chosen maximal subgroups* g (ρ) *of elements of* g *with the respective ranks* ρ.

The author has proved a similar theorem in the case where the ranks taken in their natural order are well ordered. Without some restriction on g a theorem of this sort cannot be proved. A fourth condition on the ranks is met in practice.

IV—If $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are elements of G with ranks at most a_0 while the sums $u = \Sigma u_i$ and $v = \Sigma v_j$ have no rank and $u + v$ has a rank, then $\rho(u + v) < a_0$.

We shall say that two elements u and v of G are in the same *rank class*

if u and v have the same rank while $u - v$ has no rank or a lesser rank. An isomorphism between two subgroups of G of elements with rank will be termed a *rank isomorphism* if corresponding non-null elements are in the same rank class.

THEOREM 1.2. *When conditions I to IV are satisfied any two maximal subgroups of elements of* G *with the same rank are rank isomorphic.*

This theorem enables us to assign type groups and type numbers to critical sets.

2. *The Space* M *and Function* F.—Let M be a metric space of points p, q, etc. Let $F(p)$ be a real single-valued function of the point p on M, with $0 \leqq F \leqq 1$. By the set $F \leqq b$ is meant the subset of points of M at which $F(p) \leqq b$. Let U be an homology class with elements which are non-bounding k-cycles u. If u is on $F \leqq b$, b will be called a *cycle bound* of u and of U. The greatest lower bound of the cycle bounds of U will be called the *cycle limit* $s(u)$ of U and of the elements u of U. If U is the class of bounding k-cycles $s(u)$ will not be defined. Let G be the group of all k-cycles. With some but not all of the elements u of G we have thus associated a number $s(u)$. We term $s(u)$ the rank of u. These ranks satisfy the four rank conditions of §1. Hence the theorems of §1 hold with the present interpretation. Theorem 1.1 has here the following corollary, considerably weaker than the theorem.

COROLLARY. *The sum of the dimensions of maximal groups,* g(s), *of non-bounding* k*-cycles with the respective cycle limits* s *is at least the smaller of the two numbers alef-null and the* k*th connectivity* R_k *of* M.

Up to the present point it has been immaterial whether ordinary singular cycles are understood or Vietoris cycles. From this point on we shall refer to Vietoris cycles. See M., §2.[1] We shall now state the first of two fundamental hypotheses, that of *F-accessibility*. If Vietoris cycles are used this hypothesis is fulfilled in the ordinary variational theory. It is not in general fulfilled if ordinary singular cycles are used.[2]

Under the hypothesis of F*-accessibility any non-bounding* k*-cycle which is* ~ 0 *mod* $F < c + e$ *for each positive* e, *is homologous to a* k*-cycle on* $F \leqq c$.

A non-bounding k-cycle v whose rank is $s(v)$ and which is on $F \leqq s(v)$, will be termed *canonical*. Under the hypothesis of F-accessibility there is at least one canonical k-cycle in each non-null homology class. If the sets $F \leqq c$ are compact for each $c < 1$ the hypothesis of F-accessibility is satisfied, as we prove.

k-*Caps*. A point set A will be said to be *definitely below* a (written d-below a) if A lies on $F < a - e$ for some positive e. The phase d-mod $F < a$ shall be understood to mean mod some compact set d-below a. If u is a k-cycle on $F \leqq a$, d-mod $F < a$, an homology

$$u \sim 0 \qquad \text{(on } F \leqq a, \text{ d-mod } F < a\text{)}$$

will be called an a-*homology*. A k-cycle u, d-mod $F < a$ on $F \leqq a$, not a-homologous to zero, will be called a k-cap with cap limit a. We write $a = a(u)$. These cap-limits satisfy the four rank conditions of §1.

THEOREM 2.1. *Under the hypothesis of* F-*accessibility a canonical non-bounding* k-*cycle* u *with cycle limit* s(u) *is a* k-*cap with cap limit* s(u).

Let p be a point of M at which $F(p) = c$. The set M will be said to be *locally* F-*connected of order* m > 0 at p if corresponding to each positive constant e there exists a positive constant δ such that each singular $(m-1)$-sphere on the δ-neighborhood of p and on $F \leqq c + \delta$ bounds an m-cell of diameter less than e on $F \leqq c + e$. If each subset $F \leqq c < 1$ of M is compact and if M is locally F-connected of all orders from 1 to $m + 1$ at points of $F < 1$, then the dimension of the mth homology group of $F < 1$ is at most alef-null, and the cycle limits $s(u)$ have at most the cluster value 1.

3. *Homotopic Critical Points.*—We shall say that a continuous deformation D of a subset A of M admits a *displacement function* $\delta(e)$ on A, if whenever q precedes r on a trajectory of D and $qr > e > 0$, then $F(q) - F(r) > \delta(e)$, where $\delta(e)$ is a positive single-valued function of e. A continuous deformation of E which possesses a displacement function on each compact subset of E is termed an F-*deformation* of E. A point p will be termed *homotopically* ordinary if some neighborhood of p relative to $F \leqq F(p)$ admits an F-deformation which displaces p. A point p which is not homotopically ordinary is termed *homotopically critical*.

The function F will be said to be *upper-reducible* at p if corresponding to each constant $c > F(p)$ some neighborhood of p relative to $F \leqq c$ admits on F-deformation onto a set d-below c. A function F which is lower semi-continuous is not necessarily upper-reducible, and conversely. We have the following principal theorem.

THEOREM 3.1. *If* F *is upper-reducible at each point, each cap limit is assumed by* F *in at least one homotopic critical point.*

If then the hypothesis of F-accessibility is satisfied and F is upper-reducible each cycle limit $s(u)$ is assumed by F at some homotopic critical point. This should be contrasted with the following theorem: When the space M is compact and F is lower semi-continuous, the absolute minimum of F is assumed at some critical point. As ever $F \geqq 0$.

By the *complete critical* set ω at the level c is meant the set of all homotopic critical points at which $F = c$. Any subset σ of ω which is closed in ω and at a positive distance from $\omega - \sigma$ will be termed a *critical set*. A k-cap u with cap limit c will be said to be *associated* with σ if u is c-homologous to a k-cap on an arbitrarily small neighborhood of σ. A maximal group of k-caps associated with σ will be called the kth *type group* of σ. Any two kth type groups of σ are rank isomorphic (with the ranks the cap limits). The dimension of a kth type group of σ is termed the kth *type number* of σ. A kth type group of ω can be obtained as a direct sum of the kth type groups

of any finite set of disjunct critical sets summing to ω. We have the following theorem:

THEOREM 3.2. *If* M *is* F*-accessible and* F *is upper-reducible on* $F < 1$, *the sum of the* k*th type numbers of the respective critical sets on* $F < 1$ *is at least the smaller of the two cardinal numbers, alef-null and the* k*th connectivity of* $F < 1$.

In the special case where F is locally a function of class C'' of n coördinates, and p is a critical point at which the Hessian of F is not zero, the jth type number of p equals the Kronecker δ_k^j, where k is the number of negative characteristic roots of the Hessian of F at p.

4. *Variational Theory.*—We apply the preceding theory to the problem of finding extremals joining two points a and b of a connected space Σ with a symmetric metric pq. The space of all sensed curves joining a to b on Σ with a Frêchet distance between curves will be denoted by $\Omega(a, b)$. The space $\Omega(a, b)$ here replaces M. We begin by showing that the kth homology group of $\Omega(a, b)$ is isomorphic with that of $\Omega(a', b')$, provided Σ is arcwise connected. To define F on Ω we suppose that we have a second metric $[pq]$ defined for p and q on Σ. We do not assume that $[pq] = [qp]$. Otherwise $[pq]$ shall satisfy the usual axioms. We assume that $[pq]$ is continuous in p and q in terms of the first metric pq. The function $J(\lambda)$ shall be the length of the curve λ of $\Omega(a, b)$ defined in the usual way in terms of the second metric $[pq]$. We set

$$F(\lambda) = \frac{J(\lambda)}{1 + J(\lambda)}$$

with $F(\lambda) = 1$ when $J(\lambda)$ is infinite.

We assume that Σ is *finitely* J*-compact* in that for each fixed point p of Σ and finite constant c, the subset $[pq] \leqq c$ of Σ is compact. It follows that the hypothesis of F-accessibility is satisfied on $\Omega(a, b)$. A simple sensed curve λ joining two points p and q of Σ will be termed a *right arc* if a point r lies on λ when and only when $[pq] = [pr] + [rq]$. We assume that Σ is *locally* J*-convex* in the following sense. With each point p of Σ there shall be associated a positive number $\rho(p)$ continuous in p and such that when $q \neq p$ and $[pq] \leqq \rho(p)$, p can be joined to q on Σ by a right arc every subarc of which is a right arc. It follows that F is upper-reducible on the subspace $F < 1$. This is sufficient for our purposes. A curve h will be called a *metric extremal* provided every closed subarc of h whose J-length is sufficiently small is a right arc. Regarded as a curve each homotopic critical point of F will be called a *homotopic extremal.* We have the following fundamental theorem.

THEOREM 4.1. *Each homotopic extremal of* Ω(a, b) *is a metric extremal.*

We also show that Ω (a, b) is locally F-connected, that all cycle limits are less than 1, and that the subsets $F \leqq c < 1$ are compact. The preceding

theory is readily applied to the calculus of variations with the usual positive, and positive regular integrand in parametric form.

[1] Morse, "Functional Topology and Abstract Variational Theory," *Annals of Mathematics*, **38**, 386–449 (1937). We refer to this paper by the letter M. Complete references are given in this paper and in the fasicule to appear later.

[2] The following book will appear shortly: Seifert und Threlfall, *Variationsrechnung im Grossen. Theorie von Marston Morse*. Teubner, Berlin. This book is highly recommended. The authors begin with two axioms similar to our accessibility hypothesis, but referring to singular cycles. These axioms are satisfied when the critical values cluster at most at infinity and when the critical points are isolated. In this way the most important cases are treated in the simplest way. To obtain greater generality Vietoris cycles seem to be useful. In fact the present author has shown in 3 (following) that the accessibility hypothesis is not in general satisfied when ordinary cycles are used, even when f is of class C^n on regular analytic manifolds and when the critical values are finite in number.

[3] Morse, "Sur le calcul des variations," *Bull. Societé Mathématique de France* (1938).

ON CRITERIA CONCERNING SINGULAR INTEGERS IN CYCLOTOMIC FIELDS

By H. S. Vandiver

Department of Pure Mathematics, University of Texas

Communicated June 24, 1938

As elsewhere[1] a singular integer is defined as an integer α in the field $k(\zeta)$; $\zeta = e^{2i\pi/l}$, l an odd prime, such that $\alpha = \mathfrak{a}^l$ where $\mathfrak{a}$ is an ideal in $k(\zeta)$ which is not principal. Necessary conditions that an integer in $k(\zeta)$ be singular were given by Takagi,[2] and when the field $k(\zeta)$ is properly irregular, that is to say, when the second factor of its class number is prime to l; a necessary and sufficient condition was given by the writer.[1] Here we shall give some other necessary conditions for singular integers in any irregular cyclotomic field. Based on a result of Kummer's the writer[3] obtained the relation

$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu l/k]} \mathfrak{b}(\zeta^{r_1}) \sim 1, \tag{1}$$

that is, the ideal on the left is principal, where k is an integer $1 < k < l$; $rr_1 \equiv 1 \pmod{l}$; $[s]$ is greatest integer in s, and $\mathfrak{b}$ is any integer in $k(\zeta)$ and $\mathfrak{b}(\zeta^t)$ is obtained from $\mathfrak{b}(\zeta)$ by the substitution (ζ/ζ^t); and it follows if we assume that (Vandiver[3]) α is singular and semi-primary, then

$$\prod_{\nu=1}^{k-1} \prod_{r=1}^{[\nu l/k]} \alpha(\zeta^{r_1}) = \omega^l$$

where ω is an integer in $k(\zeta)$. As it stands this is a necessary condition that α be singular, but we shall show that this can be transformed into criteria which do not involve k. The method is an extension, and modification, of one employed previously by the writer[4] in connection with units in certain cyclic fields. We shall find it convenient to employ power characters in $k(\zeta)$. Set

$$\left(\frac{\theta}{\mathfrak{p}}\right) = \zeta^{I(\theta)}$$

and

$$\theta^{\frac{N(\mathfrak{p})-1}{l}} \equiv \left(\frac{\theta}{\mathfrak{p}}\right) \pmod{\mathfrak{p}}$$

where $\mathfrak{p}$ is any ideal in $k(\zeta)$ prime to (θ) and (l) with θ an integer in $k(\zeta)$. Also put

$$D_s = \sum_{d=1}^{l-1} d^s I(\alpha(\zeta^d))$$

and consider the sum

$$\sum_{s=0}^{l-2} \sum_{d=1}^{l-1} d_1^{l-1-s} d^s I(\alpha(\zeta^d))$$

where d_1 is one of the integers in the set $1, 2, \ldots, l-1$. This may be put in the form

$$(l-1)\, I\,(\alpha^{d_1}) + \sum_{d_1 \neq d} d_1 \frac{d_1^{l-1} - d^{l-1}}{d_1 - d} I\,(\alpha(\zeta^d))$$

whence

$$-I(\alpha(\zeta^d)) \equiv D_0 + d^{l-2} D_1 + d^{l-3} D_2 + \ldots + d D_{l-2},$$

modulo l. Applying this to (1) after setting ζ^m for ζ we have, if $\mu = (l-1)/2$,

$$\mu(k-1) D_0 m^{l-1} + \sum_r (mr_1)^{l-2} D_1 + \sum_r (mr_1)^{l-3} D_2 + \ldots + \sum_r (mr_1) D_{l-2} \equiv 0 \pmod{l}.$$

Now set $m = 1, 2, \ldots, l-1$ in turn we obtain $(l-1)$ congruences and we obtain by elimination

$$D_0 \equiv \sum r_1^{l-s} D_{s-1} \equiv 0 \pmod{l}, \qquad (2)$$

$$s = 2, 3, \ldots, l-1.$$

Using the known relation

$$\frac{(1-k^i) b_i}{k^{i-1} i} \equiv \sum r_1^{l-i} \pmod{l},$$

$$b_1 = -1/2,\ b_{2a+1} = 0,\ a > 0; \qquad b_{2a} = (-1)^{a-1} B_a,$$

the B's being the numbers of Bernoulli, $B_1 = 1/6$, $B_2 = 1/30$, etc., and letting k be a primitive root of l then (2) gives

$$D_0 \equiv b_{s+1} D_s \equiv 0 \pmod{l}, \tag{3}$$

$s = 1, 2, \ldots, l-2$. Suppose that $b_{2a} \not\equiv 0 \pmod{l}$ then (3) gives

$$D_{2a-1} \equiv 0 \pmod{l},$$

$$\left(\frac{\prod_{d=1}^{l-1} \alpha(\zeta^d) d^{2a-1}}{\mathfrak{p}} \right) = 1,$$

for any $\mathfrak{p}$ prime to l and $N(\alpha)$. Hence by a known result we have the THEOREM: *If $\alpha(\zeta)$ is a singular integer in the field $k(\zeta)$ and is also semi-primary; $\zeta = e^{2i\pi/l}$, l an odd prime and $B_a \not\equiv 0$ (mod l), $a < (l-1)/2$, then*

$$A_{2a-1} = \prod_{d=1}^{l-1} (\alpha(\zeta^d))^{d^{2a-1}} = \sigma^l \tag{4}$$

where σ is an integer in $k(\zeta)$.

If in (4) we obtain an identity in e^v by adding a suitable multiple of $(l^v - 1)/(l^r - 1)$ and differentiate $(l - 2a)$ times and set $v = 0$ we obtain

$$\left[\frac{d^{l-2a} \log \alpha(l^v)}{dv^{l-2a}} \right]_{v=0} \equiv 0 \pmod{l}$$

which gives a known criterion that $\alpha(\zeta)$ is singular.

We also note that

$$A_{2a-1} = \prod_{c=0}^{l-2} (\alpha(\zeta^{r^c}))^{r^{c(2a-1)}} \delta$$

where now r is a primitive root of l and also

$$A_{2a-1}(\zeta^r) r^{2a-1} = A_{2a-1}(\zeta) \gamma^l$$

where δ and γ are numbers in $k(\zeta)$.

Additional relations of this type hold if we restrict ourselves to properly irregular cyclotomic fields. Here for any singular integer in $k(\zeta)$, say, $\omega(\zeta)$, we have

$$\omega(\zeta)\omega(\zeta^{-1}) = \epsilon\tau^l$$

where ϵ is a unit and τ an integer in $k(\zeta)$. Treating this relation in a similar manner to that employed in connection with (1) we obtain

$$\prod_{d=1}^{l-1} (\omega(\zeta^d))^{d^{2a}} = \beta^l \delta$$

for each a in the set $1, 2, \ldots, (l-3)/2$, such that δ is a unit in $k(\zeta)$ with $\delta(\zeta^r)^{r^{2a}} = \delta(\zeta)\gamma^l$, and $B_a \equiv 0 \pmod{l}$.

[1] Vandiver, these PROCEEDINGS, **15**, 203 (1929).
[2] *Jour. für die Reine und Angewandte Mathematik*, **157**, 230–238 (1927).
[3] *Ann. Math.* (2), **21**, 78 (1919).
[4] *Trans. Amer. Math. Soc.*, **29**, 156 (1927).

GROUPS OF DEGREE n INVOLVING ONLY SUBSTITUTIONS OF LOWER DEGREES

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

Communicated July 7, 1938

Every transitive group of degree n involves at least $n-1$ substitutions of this degree but an intransitive group of degree n does not necessarily involve any substitution whose degree is as large as n. Such intransitive groups exist only when $n > 4$ and there is obviously only one such group of degree 5, viz., the group formed by a 3, 1 isomorphism between the symmetric group of degree 3 and the group of degree 2. Whenever n is odd and exceeds 3 it is clearly possible to construct similarly a group of degree n which does not involve any substitution whose degree is as large as n by dimidiating the dihedral group of degree $n-2$ and the group of degree 2. Exactly half of the substitutions of this group are of degree $n-1$ while the remaining substitutions thereof appear in the cyclic subgroup of degree $n-2$ and are either of degree zero or of degree $n-2$.

From the preceding paragraph it results that when n is odd and exceeds 3 it is always possible to construct at least one group of degree n which does not involve any substitution of this degree and has one transitive constituent of degree $n-2$. We proceed to prove that such a group cannot be constructed when n is an even number. If this were possible the transitive constituent of degree $n-2$ of such a group would involve a subgroup of index 2 containing all its substitutions of degree $n-2$ since none of its other substitutions would be of as large a degree as $n-2$. Hence all of these other substitutions would be of degree $n-3$ since this is the average number of letters in the substitutions of this constituent. As $n-3$ is supposed to be odd each of these substitutions of degree $n-3$ involves an odd cycle and an odd power of such a substitution would involve less than $n-3$ letters but would appear among the given substitutions of degree $n-3$. As this is impossible it results that *a necessary and sufficient condition*

that there exists a group of degree n > 3 *which involves no substitution of degree* n *but has a transitive constituent of degree* n $- 2$ *is that* n *is odd.*

Suppose that n is the sum of two distinct prime numbers p, q, $p > q$, and that a group has two transitive constituents of degree p and q, respectively. If $p - 1$ is divisible by q it is clearly possible to construct a group of degree n which involves no substitution of degree n by establishing a p, 1 isomorphism between a subgroup of the metacyclic group of order $p(p - 1)$ and the group of degree q. On the other hand, when $p - 1$ is not divisible by q it is impossible to construct a group of degree n which does not involve a substitution of degree n since every invariant subgroup of a transitive group of degree p involves all of its substitutions of order p and its quotient group under the entire group is cyclic and has an order which divides $p-1$.[1] For the same reason when $n = 2p$, where p is an odd prime number, it is impossible to construct a group of degree n which does not contain a substitution of degree n. Hence there results the following theorem: *When* n *is the sum of two prime numbers it is impossible to construct a group of degree* n *which has these numbers for the degrees of its transitive constituents and involves no substitution of degree* n *unless the larger of them is congruent to unity with respect to the smaller.*

If at least one of the constituent groups of a group involves no substitution whose degree is equal to the degree of this constituent then the entire group cannot involve a substitution whose degree is equal to the degree of the group. In particular, if at least one of the factor groups of a direct product involves no substitution whose degree is equal to the degree of this factor group then the entire group will obviously have the same property. Since the degree of such a factor group need not exceed 5 it results that whenever $n > 6$ there is at least one direct product of degree n which has the property that it contains no substitution whose degree is equal to the degree of this direct product. It is easy to verify that no such group exists whenever $n < 7$ and that the only group of degree 6 which contains no substitution of this degree is composed of the positive substitutions of the group of order 8 and degree 6 which has three transitive constituents.

In what precedes, two substitution groups of a given degree are said to be identical whenever they are conjugate under the symmetric group of this degree. It is often desirable to consider two such groups as distinct whenever one of them contains at least one substitution which does not appear in the other. If this is done there are 10 groups of degree 5 which have the property that none of them contains a substitution of this degree since such a group is transformed into itself by 12 substitutions on these 5 letters. Similarly there are 15 groups of degree 6 which separately do not involve a substitution of this degree since each of them is transformed into itself by 48 substitutions on these 6 letters. In what follows we shall employ the latter of these two methods of enumerating substitution groups unless the

contrary is stated. There are therefore 210 groups on 7 letters, each being of degree 5, which separately have the property that none of them involves a substitution which is equal to the degree of the group, and there are 105 groups on 7 letters, each being of degree 6, which separately have the same property.

It was noted above that whenever $n > 6$ there is always at least one direct product of degree n which does not contain a substitution of this degree. We proceed to prove that there is then also at least one group of this degree involving no substitution of this degree but in which every constituent group contains at least one substitution whose degree is equal to the degree of this constituent. When n is odd such a group can obviously be constructed by establishing a dimidiation between the dihedral group of degree $n - 2$ and the group of degree 2 on the remaining letters. When n is even the dihedral group of degree $n - 4$ involves $(n - 4)/2$ substitutions of degree $n - 6$ and the same number of substitutions of degree $n - 4$ besides the cyclic subgroup of this order. Hence we can establish an isomorphism between this dihedral group and the intransitive group of order 4 on the remaining letters so that no substitution has a degree which exceeds $n - 2$ and all the substitutions are of this degree except those in the cyclic subgroup of order $(n - 4)/2$. As every constituent group of this group involves at least one substitution whose degree is equal to the degree of this constituent there results the following theorem: *There is at least one group of every degree* n > 6 *which has the property that it does not involve any substitution of degree* n *and that each of its constituent groups involves at least one substitution whose degree is equal to the degree of this constituent.*

To obtain an expression for the number of the groups in each of the two categories noted in the preceding paragraph it may be observed that when n is odd the given cyclic group of degree $n - 2$ can be chosen in $n(n - 1)\ldots 3/(n - 2)$ ways and one such group corresponds to each of these cyclic subgroups. When n is even the given cyclic group of degree $n - 4$ can be selected in $n(n - 1)\ldots.5/(n - 4)$ ways but for each of these cyclic subgroups there are three of the required groups. Hence the number of the possible groups which satisfy the given condition in this case is $3n(n - 1)\ldots.5/(n - 4)$. When n is odd the given number of these groups exceeds $(n - 1)!/2$. This is of interest in connection with the question of the relative number of substitutions and subgroups in the symmetric group of degree n.

If a group of degree n is constructed by establishing a simple isomorphism between symmetric groups of degree m it necessarily involves substitutions of degree n. To prove this theorem it may first be noted that when $m \neq 6$ then transpositions of these various symmetric groups correspond to each other in this simple isomorphism in view of the fact that when $m > 6$ a transposition is transformed into itself by more substitu-

tions of the symmetric group than any other substitution of order 2 contained therein. Hence these transpositions correspond to each other in such a simple isomorphism except possibly when $m = 6$. In this particular case the positive substitutions of order 4 of such a symmetric group correspond to each other and hence the resulting group contains substitutions of degree n and does not require further consideration here. In all other cases a transposition and a cyclic substitution of degree $m - 1$ which involves one and only one letter of this transposition generate the symmetric group of degree m. The latter therefore corresponds also to such a substitution. As the product of these two substitutions is cyclic and of degree m it results that every group of degree n which is formed by a simple isomorphism between symmetric groups involves at least one substitution of degree n.

Suppose that a group has the property that all its operators appear in two sets of conjugate subgroups which are separately neither invariant nor contain any invariant subgroup of the entire group besides the identity. Such a group can be represented as a transitive substitution group with respect to each of these sets of conjugate subgroups and these two transitive representations of the group can be placed into a simple isomorphism in such a way that the resulting group of degree n contains no substitution of degree n. As an illustration of such a group we cite the symmetric group of degree 4 which may be represented as a transitive group of degree 4 with respect to its subgroups of order 6 and as a transitive group of degree 6 with respect to its cyclic subgroups of order 4. The two groups thus obtained can be placed either into a simple isomorphism or into a 4, 4 correspondence so as to obtain a group of degree 10 which involves no substitution whose degree exceeds 9. In the alternating group of degree 5 all the substitutions appear either in the conjugate subgroups of order 12 or in the conjugate subgroups of orders 5 or 10. Simple isomorphisms between these groups can therefore be established in more than one way so as to obtain groups containing no substitution whose degree is equal to the degree of the group.

[1] G. A. Miller, *Bull. Amer. Math. Soc.*, **4**, 141 (1893).

ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

BY J. L. WALSH AND W. SEIDEL

DEPARTMENTS OF MATHEMATICS, HARVARD UNIVERSITY AND UNIVERSITY OF ROCHESTER

Communicated July 5, 1938

Various results concerning the order of growth of the first and higher derivatives of univalent and of bounded functions analytic in the unit circle are known. Among these we mention Koebe's distortion theorem (Verzerrungssatz) in the univalent case and Schwarz's Lemma and the results of O. Szász[1] in the bounded case. A consequence of these results for a function $f(z)$ analytic in $|z| < 1$ is $|f'(z)| = 0\left(\frac{1}{(1 - |z|)^3}\right)$ in the case that $f(z)$ is univalent and $|f^{(n)}(z)| = 0\left(\frac{1}{(1 - |z|)^n}\right)$ in the case that $f(z)$ is bounded. These relations, however, give no information as to the conditions under which $|f^{(n)}(z_k)|(1 - |z_k|)^n$ tends to zero for a given sequence of points $\{z_k\}$ in the unit circle. In the univalent case an answer to this question is contained in the following result due to J. E. Littlewood and A. J. Macintyre:[2] *Let* f(z) *be analytic and univalent in* $|z| < 1$ *and let it omit there the value* ω. *Then in* $|z| < 1$ *the following inequality is satisfied:*

$$|f'(z)|(1 - |z|^2) \leqq 4\,|\,\omega - f(z)\,|.$$

The purpose of the present note is to summarize the principal results recently obtained by the authors in the study of the relation

$$|f^{(n)}(z_k)|(1 - |z_k|)^n \to 0 \text{ as } k \to \infty, \text{ where } |z_k| < 1,$$

in the case that the analytic function $f(z)$ is univalent, or is bounded, or omits two values in $|z| < 1$. The method involves primarily the systematic and detailed study of the function $\varphi_k(\zeta) = f\left(\frac{z_k + \zeta}{1 + \bar{z}_k\zeta}\right)$. The detailed exposition will be published at a later date.

Let $w = f(z)$ be analytic in $|z| < 1$ and let R denote the Riemann configuration over the w-plane onto which this function maps the region $|z| < 1$. Let w_0 be an arbitrary point of R. Then the radius of the largest smooth circle (boundary not included) with center at w_0 and wholly contained in R is called the *radius of univalence of* R *at* w_0 and will be denoted by $D_1(w_0)$. The following theorem can be easily proved:

Let f(z) *be regular and univalent in* $|z| < 1$, z_0 *any point of* $|z| < 1$, *and* $w_0 = f(z_0)$. *Then*

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq 4D_1(w_0). \qquad (1)$$

Each of these inequalities is sharp and the second one is another form of the Littlewood-Macintyre inequality. An immediate consequence of this theorem is the corollary:

Let $f(z)$ *be regular and univalent in* $|z| < 1$, z_n *any sequence of points in* $|z| < 1$, *and* $w_n = f(z_n)$. *Then a necessary and sufficient condition that*

$$\lim_{n \to \infty} |f'(z_n)|(1 - |z_n|) = 0 \tag{2}$$

is that

$$\lim_{n \to \infty} D_1(w_n) = 0, \tag{3}$$

and a necessary and sufficient condition that $|f'(z_n)|(1 - |z_n|)$ *be bounded is that* $D_1(w_n)$ *be bounded.*

From this theorem it follows for a univalent and bounded function $f(z)$ that $|f'(z)| = o\left(\frac{1}{1 - |z|}\right)$ uniformly in the circle $|z| < 1$ as $|z| \to 1$. In this connection the following theorem is appropriate:

Let $f(z)$ *be regular and univalent in the circle* $|z| < 1$. *Then*

$$\lim_{z \to e^{i\alpha}} |f'(z)|(1 - |z|)^{1/2} = 0$$

for all points $e^{i\alpha}$ *of the circumference* $|z| = 1$ *with the exception of at most a set of Lebesgue measure zero, where* z *in the above limit is taken in any angle less than* π *with vertex in* $e^{i\alpha}$ *and bisected by the corresponding radius.*

For univalent functions the second part of inequality (1) admits of extensions to all higher derivatives, for instance

$$|f^{(n)}(z_0)|(1 - |z_0|^2)^n \leqq 4 \cdot e \cdot n!\,(|z_0| + n)(1 + |z_0|)^{n-2} D_1(w_0).$$

For $n = 2$ or 3 the factor e in the right side may be omitted; the resulting inequalities are sharp. Obviously, $D_1(w_k) \to 0$, $w_k = f(z_k)$, implies

$$|f^{(n)}(z_k)|(1 - |z_k|)^n \to 0. \tag{4}$$

Examples can be constructed to show that the limit in (2) and in (4) can be approached arbitrarily slowly even if $f(z)$ is univalent and continuous in $|z| \leqq 1$.

The situation is entirely analogous for bounded functions:

Let $f(z)$ *be regular and bounded in* $|z| < 1$:

$$|f(z)| \leqq M,$$

let $\{z_n\}$ *be any sequence of points in* $|z| < 1$, *and let* $w_n = f(z_n)$. *Then a necessary and sufficient condition for* (2) *is* (3).

Indeed, for any z_0 $(|z_0| < 1)$ *we have*

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq \sqrt{8MD_1(w_0)}, \tag{5}$$

where $w_0 = f(z_0)$.

The situation for the higher derivatives of bounded functions is somewhat more complicated than in the univalent case. We need the following definitions:

Let R be a Riemann surface (configuration) over the w-plane, let w_0 be any point of R not a branch point of order greater than $p-1$. Suppose that C_p is a simply connected region on R which contains w_0 in its interior, lies over the circle $|w - w_0| < \rho$, and covers this circle precisely p times. We call such a region a p-*sheeted circle of center* w_0 *and radius* ρ; the value $\rho = \infty$ is not excluded. *The radius of* p-*valence* $D_p(w_0)$ *of a Riemann surface* R *at a point* w_0 *belonging to* R *is defined as follows:*

(a) For $p = 1$, $D_p(w_0) = D_1(w_0)$.

(b) If w_0 is a branch point of order greater than $p-1$ ($p > 1$), then $D_p(w_0) = 0$.

(c) For any other point w_0, the number $D_p(w_0)$ is the radius of the largest p-sheeted circle with center w_0 contained in R if such a circle exists, and is otherwise $D_{p-1}(w_0)$.

With this definition we prove:

Let $\{f_n(z)\}$ *be a sequence of functions analytic in the unit circle* $|z| < 1$ *and converging uniformly in every closed subregion of* $|z| < 1$ *to an analytic function* $f(z)$. *Let* z_0 *be any point in the circle* $|z| < 1$ *and set* $w_n = f_n(z_0)$, $w_0 = f(z_0)$. *Let* $D_p(w_n)$ *and* $D_p(w_0)$ *pertain to the images of* $|z| < 1$ *by the maps* $w = f_n(z)$ *and* $w = f(z)$, *respectively. Then*

$$\lim_{n \to \infty} D_p(w_n) = D_p(w_0).$$

As an analogue of the inequalities (5) we obtain:

Let $f(z)$ *be regular and bounded in* $|z| < 1$: $|f(z)| \leqq M$, *let* z_0 *be any point of* $|z| < 1$ *and* $w_0 = f(z_0)$. *Let* p *be a positive integer. Then there exist two positive constants* λ_p *depending only on* p *and* Λ_p *depending on* p *and* M *such that we have*

$$\lambda_p \cdot D_p(w_0) \leqq \sum_{k=1}^{p} \frac{1}{k!} \left| \sum_{\nu=0}^{k-1} (-1)^{\nu} \cdot \nu! \binom{k}{\nu} \binom{k-1}{\nu} \bar{z}_0^{\nu} \cdot (1 - |z_0|^2)^{k-\nu} \cdot f^{(k-\nu)}(z_0) \right| \leqq \Lambda_p [D_p(w_0)]^{2^{-p}}. \quad (6)$$

Consequently, for any sequence $\{z_n\}(|z_n| < 1)$, *a necessary and sufficient condition for*

$$\lim_{n \to \infty} f^{(k)}(z_n)|(1 - |z_n|)^k = 0 \qquad (k = 1, 2, \ldots, p)$$

with $w_n = f(z_n)$, *is that*

$$\lim_{n \to \infty} D_p(w_n) = 0.$$

The last part of this theorem may also be inferred from the preceding theorem. Constants λ_p and Λ_p for which (6) holds can be explicitly determined.

Schottky's theorem enables us to extend the above results for bounded functions to the case of functions $f(z)$ omitting two values provided the sequence $w_n = f(z_n)$ is bounded. The case $|w_n| \to \infty$ can be treated by other methods.

[1] O. Szász, *Math Zeitschrift*, **8**, 303–309 (1920).

[2] J. E. Littlewood, *Proc. London Math. Soc.*, **23**, 507 (1924); A. J. Macintyre, *Jour. London Math. Soc.*, **11**, 7–11 (1936).

DIFFERENTIAL CALCULUS IN LINEAR TOPOLOGICAL SPACES[1]

BY A. D. MICHAL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated July 9, 1938

1. *Introduction.*—The most valuable definitions of differentials of functions in the classical differential calculi of finite as well as of infinite dimensional spaces are those that give the differential as a "first order approximation" to the difference. In this paper we give a definition of such a differential for functions whose arguments are in a linear topological space T_1 and whose values are in a linear topological space T_2, not necessarily the same[2] as T_1. Some of the fundamental properties of this differential are given as well as the properties of other related topological differentials.

We wish to emphasize here the fact that the spaces T_1 and T_2 are not necessarily metric—not even metrizable—and that the differential calculus in linear topological spaces has important applications to general differential geometry, general dynamics and general continuous group theory.

2. *Topological* M-*Differential.*—By a linear topological space we shall mean an abstract linear space with a Hausdorff topology in which the functions $x + y$ and αx are respectively continuous functions of both variables.

Let T_1 and T_2 be any two linear topological spaces. A function $l(x)$ on T_1 to T_2 is termed *linear* if it is additive and continuous—hence homogeneous of degree one.

DEFINITION OF *M*-DIFFERENTIAL.[3] *Let* $f(x)$ *be a function with values in* T_2 *and defined on a Hausdorff neighborhood* S_x, *of* $x_0 \epsilon T_1$. *The function* $f(x)$ *will be said to be* M-*differentiable at* $x = x_0$ *and* $f(x_0; \delta x)$ *will be called an* M-*differential of* $f(x)$ *at* $x = x_0$ *with increment* δx *if*

(1) *there exists a linear function* $f(x_0; \delta x)$ *of* δx *with arguments in* T_1 *and values in* T_2

(2) *there exists a function* $\epsilon(x_0, x_1, x_2)$ *with arguments in* T_1 *and values in* T_2 *such that*

$$\text{(a)}\ \epsilon(x_0, 0, x) = 0 \text{ for all } x \epsilon T_1$$
$$\text{(b)}\ \epsilon(x_0, x_1, \lambda x_2) = \lambda \epsilon(x_0, x_1, x_2)$$

for all $\lambda > 0$, *for all* x_1 *in some Hausdorff neighborhood of* $0 \epsilon T_1$, *and for all* $x_2 \epsilon T_1$

$$\text{(c)}\ \epsilon(x_0, x_1, x_2)$$

is continuous in (x_1, x_2) *at* $x_1 = 0$, $x_2 = x_2$ *for all* $x_2 \epsilon T_1$.

(3) *there exists some Hausdorff neighborhood* S_0' *of* $0 \epsilon T_1$ *such that for all* $\delta x \epsilon S_0'$

$$f(x_0 + \delta x) - f(x_0) - f(x_0; \delta x) = \epsilon(x_0, \delta x, \delta x).$$

THEOREM 1. *If an* M*-differential of* $f(x)$ *at* $x = x_0$ *exists, then it is unique and* $f(x)$ *is continuous at* $x = x_0$.

THEOREM 2. *If* $f_1(x)$ *and* $f_2(x)$ *are* M*-differentiable at* $x = x_0$ *then* $f_3(x) = \alpha f_1(x) + \beta f_2(x)$ *is* M*-differentiable at* $x = x_0$ *and*

$$f_3(x_0; \delta x) = \alpha f_1(x_0; \delta x) + \beta f_2(x_0; \delta x).$$

THEOREM 3. *Let* T_3 *be a third linear topological space. If* $f(x)$ *on* $S_{x_0} \subset T_1$ *to* T_2 *is* M*-differentiable at* $x = x_0$ *and if* $\phi(y)$ *on* $f(S_x)$ *to* T_3 *is* M*-differentiable at* $y_0 = f(x_0)$, *then* $\psi(x) = \phi(f(x))$ *is* M*-differentiable at* $x = x_0$ *and*

$$\psi(x_0; \delta x) = \phi(f(x_0); f(x_0; \delta x)).$$

3. OTHER DIFFERENTIALS AND THEIR RELATION TO THE *M*-DIFFERENTIAL. DEFINITION OF *G*-DIFFERENTIAL. *Let* $f(x)$ *be a function defined on a Hausdorff neighborhood* S_x, *of* $x_0 \epsilon T_1$ *and with values in* T_2. *We shall say that* $f(x)$ *is* G*-differentiable at* $x = x_0$ *and* $f(x_0, \delta x)$ *is its* G*-differential at* $x = x_0$ *with increment* δx *if for any chosen* $\delta x \epsilon T_1$:

Given any Hausdorff neighborhood V_0 *of* $0 \epsilon T_2$ *there exists a* $\delta > 0$ *such that*[4]

$$\frac{f(x_0 + \lambda \delta x) - f(x_0)}{\lambda} \epsilon f(x_0, \delta x) + V_0$$

for each λ *satisfying* $0 < |\lambda| < \delta$.

THEOREM 4. *If* $f(x)$ *is* M*-differentiable at* $x = x_0$, *then* $f(x)$ *is* G*-differentiable at* $x = x_0$ *and the two differentials are equal.*

DEFINITION OF *H M*-DIFFERENTIAL.[5] *Let* $f(x)$ *be a function with values in* T_2 *and defined on a Hausdorff neighborhood* S_{x_0} *of* $x_0 \epsilon T_1$, *and let* $x(\lambda)$ *be any chosen function of a real variable* λ *with values in* S_{x_0} *and possessing a derivative* $\frac{dx(\lambda)}{d\lambda}$ *at any chosen* $\lambda = \lambda_0$. *Write* $x_0 = x(\lambda_0)$. *The function* $f(x)$ *will be said to be* H M*-differentiable at* $x = x_0$ *with* $f(x_0; \delta x)$ *as its* H M*-dif-*

ferential at $x = x_0$ *if there exists a linear function* $f(x_0:\delta x)$ *of* δx *having arguments in* T_1 *and values in* T_2 *such that for every admissible* $x(\lambda)$:

$$\frac{d}{d\lambda} f(x(\lambda)) \text{ exists at } \lambda = \lambda_0 \tag{1}$$

$$\frac{d}{d\lambda} f(x(\lambda)) = f\left(x_0 : \frac{dx(\lambda)}{d\lambda}\right) \text{ for } \lambda = \lambda_0. \tag{2}$$

THEOREM 5. *If* $f(x)$ *is* M*-differentiable at* $x = x_0$, *then* $f(x)$ *is* H M*-differentiable at* $x = x_0$ *and the two differentials are equal.*

THEOREM 6. *If the linear topological spaces* T_1 *and* T_2 *are complete linear normed spaces (Banach spaces) and if* $f(x)$ *is Fréchet differentiable*[6] *at* $x = x_0$, *then* $f(x)$ *is* M*-differentiable at* $x = x_0$ *and the two differentials are equal.*

THEOREM 7. *If the linear topological spaces* T_1 *and* T_2 *are finite dimensional arithmetic spaces and if* $f(x)$ *is differentiable at* $x = x_0$ *in the Stolz-Young-Fréchet sense, then* $f(x)$ *is* M*-differentiable at* $x = x_0$ *and the differentials are equal. Conversely if* $f(x)$ *is* M*-differentiable at* $x = x_0$, *then it is differentiable in the Stolz-Young-Fréchet sense.*

[1] Presented to the American Mathematical Society, April 9, 1938.

[2] In case T_1 is a special linear topological space and T_2 is the same space as T_1, a certain topological differential was defined by Michal and Paxson. It is still an open question whether the differentiability theorem on the composition of functions is valid for the Michal-Paxson differential. See Michal, A. D., and Paxson, E. W.: (1) "La Différentielle dans les Espaces Linéaires Abstraits avec une Topologie," *Comptes Rendus, Paris,* **202,** 1741–1743 (1936); (2) "The Differential in Abstract Linear Spaces with a Topology," *Comptes Rendus de la Soc. de Sc. de Varsovie,* XXIX, 106–121 (1936).

[3] Another interesting type of differential can be defined by merely changing the equality relation in condition (3) of the definition for an *M*-differential into a set inclusion relation.

[4] By $f(x_0, \delta x) + V_0$ we mean the *set* of all elements $f(x_0, \delta x) + y$ as y ranges over the set V_0.

[5] A modified *H M*-differential is obtained if λ_0 is always taken to be $\lambda_0 = 0$. This modified *H M*-differential is itself the abstraction of a differential for a function space studied recently by Fréchet. See page 244 of Fréchet, M., "Sur la Notion de Différentielle," *Journal de Math. Pures et Appl.,* **16,** 233–250 (1937).

[6] Fréchet, M., "La Notion de Différentielle dans l'Analyse Générale," *Annales Ecole Norm.,* **XLII,** 293–323 (1925).

MINIMAL SURFACES OF HIGHER TOPOLOGICAL STRUCTURE

BY JESSE DOUGLAS

BROOKLYN, N. Y.

Communicated June 30, 1938

1. About two years ago, the author published a solution of the Plateau problem in the following general topological form.

Given:

(1) k *contours* Γ *in the form of Jordan curves in* n*-dimensional euclidean space, each of assigned form, position and sense of description;*

(2) *a prescribed genus* h *or topological characteristic* r;[1]

(3) *either character of orientability, i.e., two-sided or one-sided;*

to determine a minimal surface M *corresponding to these data.*

This problem had been formulated a number of years before by the author[2] and the important special cases $k = 1, r = 0$; $k = 2, r = 0$; $k = 1, r = 1$ (Möbius strip) solved during the intervening time.[3] Indeed, the formulas of the paper on the two-contour case were explicitly developed in a form valid for the general case.[4]

The publication referred to at the beginning was in the form of two papers, the first of which stated the results, while the second gave proofs and other details.[5]

Subsequently, an alternative method of solution of the same problem was announced by R. Courant,[6] who, the following year, published the details for the case of genus zero.[7] As far as the case of one contour and a simply-connected minimal surface is concerned, Courant's method was presented independently by L. Tonelli,[8] who remarks that it coincides substantially with the one used by the author throughout his work on the Plateau problem.

2. Early next year, the *Annals of Mathematics* will publish a comprehensive paper by the author with the same title as the present note. This paper first reviews and elaborates the essential parts of the author's previous papers on the subject, and then provides various supplementary considerations, relating particularly to the case of genus $h > 0$.

The purpose of the note at hand is to present a summary of some of these more novel features of the detailed exposition contained in the forthcoming *Annals* paper.[9]

3. To typify immediately the distinction between the cases $h = 0$ and $h > 0$, we write down the respective formulas for the Green's function of the basic Riemann surface R, namely,[10]

$$h = 0: \quad G(z, w) = -\Re\Omega(z;\ w, \bar{w}); \tag{1}$$

$$h > 0: \quad G(z, w) = -\Re\{\Omega(z;\ w, \bar{w}) - T_{\alpha\beta}[v_\alpha(z) - v_{\alpha'}(z)]\,[v_\beta(w,\ \bar{w}) - v_{\beta'}(w,\ \bar{w})]\}. \tag{2}$$

Here, $\Omega(z; w, \overline{w})$ denotes the *normal integral of the third kind*[11] with logarithmic singularities at the conjugate or symmetric points $w, \overline{w}$. The other notation and general background will appear in the course of the explanations that follow.

The properties which define $G(z, w)$ uniquely are: (I) *uniform* on R, (II) *harmonic* on R, (III) logarithmic singularity at $z = w$, with principal term $-\log|z - w|$, (IV) $G(z, w) = 0$ when z is on the boundary of R.

In the case $h = 0$, formula (1) has all these properties; but when $h > 0$, the property of *uniformity* is lost, since what we shall call *alter-symmetric* circuits[12] then appear on R, with respect to which circuits G has, in general, non-vanishing periods. The purpose of the complementary terms in (2) is precisely *to neutralize these periods and restore uniformity*, while still preserving the other essential properties of Green's function.

4. As standard domain on which to represent the required minimal surface M, we employ a "semi Riemann surface" R. This is either of the two conjugate halves of a complete Riemann surface $\mathfrak{R}$ which is supposed to be *symmetric*, i.e., to have an inverse conformal transformation T into itself which associates the points of $\mathfrak{R}$ in pairs $P, \overline{P}$, called *conjugate* or *symmetric*. Thus $T^2 = 1$. $\mathfrak{R}$ always belongs as Riemann surface to a class of birationally equivalent real algebraic curves $\mathfrak{A}$: $P(x, y) = 0$, and T may be interpreted as the interchange of conjugate complex points $(x, y), (\overline{x}, \overline{y})$ of $\mathfrak{A}$. Then the real branches C of $\mathfrak{A}$ remain, point by point fixed under T. They are said to constitute the *curves of transition* of $\mathfrak{R}$ and may separate that surface or not. According as the former or the latter is the case, $\mathfrak{R}$ is said to be of the *first kind* or *second kind*. These two kinds of Riemann surface correspond, respectively, to two-sided or to one-sided minimal surfaces $M, \overline{M}$. For if we regard a pair of symmetric points $P, \overline{P}$ as a single geometric element, then $\mathfrak{R}$ becomes a manifold R, which is one-sided or two-sided according as it is or is not possible to pass continuously from P to $\overline{P}$ without traversing the boundaries of R; and these boundaries are constituted by the transition curves C of the original surface $\mathfrak{R}$.

In the case of separation of $\mathfrak{R}$ by C, either half may be identified as R; and R, previously termed a semi Riemann surface, may also be described simply as a Riemann surface, except that R has the boundaries C, while $\mathfrak{R}$ is closed. We require R to have exactly the topological structure prescribed for M, i.e., genus h and k boundaries. The genus of the complete surface $\mathfrak{R}$, on the other hand, is

$$p = 2h + k - 1 = r + k - 1 \tag{3}$$

The case of one-sided, or non-orientable, minimal surfaces $\overline{M}$ can be dealt with by means of a two-sided, or orientable, *covering surface*, in the form of a Riemann surface $\mathfrak{R}$ of the second kind, which is in two-one corre-

spondence with $\overline{M}$. That is, each pair of symmetric points P, $\overline{P}$ of $\mathfrak{R}$ correspond to the same point Q of $\overline{M}$. We may say that symmetric elements of $\mathfrak{R}$ about P, $\overline{P}$ correspond to elements of $\overline{M}$ about the point Q which are geometrically coincident but antipodal, i.e., belong to opposite senses of the normal to $\overline{M}$.

Sufficiently typical is the case of the Möbius strip, where the covering surface of $\overline{M}$ may be arranged as an orientable doubly-connected minimal surface M with two boundaries, which coincide with the opposite senses of description $+\Gamma$, $-\Gamma$ of the given contour Γ.[13]

5. In (1) and (2), the normal integral of third kind can be represented explicitly in terms of Riemann's multiple θ-functions as follows:

$$\Omega(z;\ w, \overline{w}) = \log \frac{\theta(v(z, w);\ \tau)}{\theta(v(z, \overline{w});\ \tau)}. \tag{4}$$

Here $v(z, w)$ denotes the system of normal abelian integrals of the first kind, from the lower limit w to the upper limit z:

$$v_j(z, w) = v_j(z) - v_j(w) \qquad (j, k = 1, 2, \ldots, p). \tag{5}$$

τ denotes the matrix of periods of v. The multiple θ-series is defined explicitly by[14]

$$\theta(u;\ \tau) = \Sigma \exp\ [2\pi i u_j(n_j + \tfrac{1}{2}\alpha_j) + \pi i \tau_{jk}(n_j + \tfrac{1}{2}\alpha_j)(n_k + \tfrac{1}{2}\alpha_k) + \pi i \beta_j(n_j + \tfrac{1}{2}\alpha_j)], \tag{6}$$

where the summation convention as to repeated indices is applied to j, k and Σ denotes a multiple summation as $n_1, n_2, \ldots, n_p$ range over all integers from $-\infty$ to $+\infty$. The *characteristic* $\frac{1}{2}\beta$, $\frac{1}{2}\alpha$, consisting of $2p$ half-integers, usually 0 or $1/2$, is supposed to be *odd*, i.e.,

$$\beta\alpha = \beta_1\alpha_1 + \beta_2\alpha_2 + \ldots + \beta_p\alpha_p = \text{odd integer.}[15] \tag{7}$$

6. The Riemann surface $\mathfrak{R}$ of genus p has $2p$ independent circuits: p of the *first system* A_j, and p of the *second system* B_j. These can be arranged in such a way that the inverse conformal automorphism T of $\mathfrak{R}$ relates the indices j in pairs j, j' so that we have

$$TA_j = -A_{j'}, \qquad TB_j = B_{j'}. \tag{8}$$

We shall call such indices j, j' *symmetric* to each other, and shall term the index j and the corresponding circuits A_j, B_j *self-symmetric* or *alter-symmetric* according as $j = j'$ or $j \neq j'$.

The essential feature of the case $h = 0$ *is that then all indices and circuits are self-symmetric.*

If, on the contrary, $h > 0$, then there are h alter-symmetric indices, belonging to the circuits A_α, B_α about the h handles of the semi-surface R.

These have corresponding symmetric circuits $A_{\alpha'}$, $B_{\alpha'}$ on the conjugate semi-surface R'.

As notation, we use $\alpha, \beta, \lambda, \mu$ to represent indices with the range

$$\alpha, \beta, \lambda, \mu = 1, 2, \ldots, h \tag{9}$$

corresponding to the alter-symmetric circuits on R, and

$$\alpha', \beta', \lambda', \mu' = h + 1, h + 2, \ldots, 2h \tag{10}$$

to represent the *respective* symmetric indices belonging to circuits on R'.

The quantities

$$t_{\alpha\beta} = \frac{1}{2\pi i} (\tau_{\alpha\beta} - \tau_{\alpha'\beta} - \tau_{\alpha\beta'} + \tau_{\alpha'\beta'}) \tag{11}$$

play an important rôle. . (Since $\tau_{\alpha\beta} = \tau_{\beta\alpha}$, we also have $t_{\alpha\beta} = t_{\beta\alpha}$.)

Using (8), we easily show that

$$v_{j'}(w) = -\overline{v_j(\overline{w})}, \tag{12}$$

$$\tau_{j'k'} = -\bar{\tau}_{jk}, \tag{13}$$

where the bar denotes the conjugate complex quantity. It follows from (13) that $t_{\alpha\beta}$ is *real*, in fact

$$t_{\alpha\beta} = \Re \frac{1}{\pi i} (\tau_{\alpha\beta} - \tau_{\alpha'\beta}). \tag{14}$$

The determinant $|t_{\alpha\beta}|$ is readily seen to be non-vanishing, and hence the reciprocal matrix $T_{\alpha\beta}$ can be formed, where

$$T_{\alpha\lambda} t_{\lambda\mu} = \delta_{\alpha\mu} = \begin{cases} 1 \text{ if } \alpha = \mu, \\ 0 \text{ if } \alpha \neq \mu. \end{cases} \tag{15}$$

By the symmetry of $t_{\alpha\beta}$, we also have evidently $T_{\alpha\beta} = T_{\beta\alpha}$.

These are exactly the quantities $T_{\alpha\beta}$ which appear in the complementary terms of the general formula (2) for Green's function. Repetition of indices denotes summation over the range (9). If $h = 0$, *these complementary terms are simply not present*, and the expression for Green's function reverts to (1).

7. The formula (2) has the characteristic form for Green's function:

$$G(z, w) = -\Re\{S(z, w) - S(z, \overline{w})\}. \tag{16}$$

In the present instance, we have explicitly, by reference to (2) and (4),

$$S(z, w) = \log \theta(v(z, w); \tau) - T_{\alpha\beta}[v_\alpha(z) - v_{\alpha'}(z)][v_\beta(w) - v_{\beta'}(w)]. \tag{17}$$

We have found it convenient to introduce also

$$Z(z, w) = \frac{\partial}{\partial z} S(z, w), \qquad P(z, w) = \frac{\partial^2}{\partial z \partial w} S(z, w). \tag{18}$$

The notation S, Z, P is intended to recall the elliptic functions $\log \sigma$, ζ, $\wp$, to which the former reduce essentially when the Riemann surface $\mathfrak{R}$ is of genus $p = 1$.

In the case of k arbitrary, *but* $h = 0$, the complementary terms disappear, as has been said, from the formula (17); we then have simply

$$S(z, w) = \log \theta(v(z, w);\ \tau). \tag{19}$$

By the second defining relation (18), with reference to the later explained notation (35), this gives

$$P(z, w) = \wp_{jk}(v(z, w))v_j'(z)v_k'(w), \tag{20}$$

where we leave implicit the dependence of our functions on the periods τ.

But if $h > 0$, we have, with complementary terms,

$$P(z, w) = \wp_{jk}(v(z, w))v_j'(z)v_k'(w) - T_{\alpha\beta}[v_\alpha'(z) - v_{\alpha'}'(z)][v_\beta'(w) - v_{\beta'}'(w)] \tag{21}$$

8. Let us denote by $\mathbf{H}(u, v)$, with the components $H_i(u, v)$ $(i = 1, 2, \ldots n)$, the harmonic vector function on R with the boundary values $\mathbf{g}(z)$ on C.[16] We can always express $\mathbf{H}(u, v)$ in the form

$$\mathbf{H}(u, v) = \Re\mathbf{F}(w), \qquad w = u + iv. \tag{22}$$

Then the "vector Dirichlet integral"

$$D(\mathbf{H}) = \frac{1}{2}\int\int_R (\mathbf{H}_u^2 + \mathbf{H}_v^2)\, du\, dv = \frac{1}{2}\int\int_R \sum_{i=1}^{n} \left[\left(\frac{\partial H_i}{\partial u}\right)^2 + \left(\frac{\partial H_i}{\partial v}\right)^2\right] du\, dv, \tag{22'}$$

depends for its value only on $\mathbf{g}$ and R: it is a *functional* of $(\mathbf{g}, R)$, which we denote by $A(\mathbf{g}, R)$. Upon this functional our entire method of solution is based; the required minimal surface M is determined by means of the principle

$$A(\mathbf{g}, R) = \min, \tag{22''}$$

where the range of the argument $(\mathbf{g}, R)$ consists of all Riemann surfaces R having the topological form prescribed for M, together with all parametric representations $\mathbf{g}$ of the given contours Γ on the boundaries C of R.

In "*Two Contours*" we established the following two fundamental formulas, governing our treatment of the Plateau problem for the *general topological case:*

$$A(\mathbf{g}, R) = D(\mathbf{H}) = \frac{1}{4\pi}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2\, P(z, \zeta)\, dz\, d\zeta, \tag{23}$$

$$\mathbf{F}'^2(w) = \sum_{i=1}^{n} F_i'^2(w) = \frac{1}{2\pi^2}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 P(z, w)P(\zeta, w)dz\,d\zeta. \quad (24)$$

In the case $h = 0$, these are explicitly, by reference to (20),

$$A(\mathbf{g}, R) = \frac{1}{4\pi}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2\, \wp_{jk}(v(z, \zeta))v_j'(z)v_k'(\zeta)\, dz\, d\zeta, \quad (25)$$

$$\mathbf{F}'^2(w) = \frac{1}{2\pi^2}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2\, \wp_{jl}(v(z, w))\wp_{km}(v(\zeta, w))v_j'(z)v_k'(\zeta) v_l'(w)v_m'(w)dz\, d\zeta. \quad (26)$$

In the case $h > 0$, as is seen by reference to (21), these formulas acquire complementary terms $\mathfrak{C}$ as follows:

$$\mathfrak{C}[A(\mathbf{g}, R)] = -\frac{1}{4\pi}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2\, T_{\alpha\beta}[v_\alpha'(z) - v_{\alpha'}'(z)] \times [v_\beta'(\zeta) - v_{\beta'}'(\zeta)]dz\, d\zeta, \quad (27)$$

$$\begin{aligned}\mathfrak{C}[\mathbf{F}'^2(w)] = &-\frac{1}{2\pi^2}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 T_{\alpha\beta}[v_\alpha'(\zeta) - v_{\alpha'}'(\zeta)] \times \\ &\qquad [v_\beta'(w) - v_{\beta'}'(w)]\wp_{jk}(v(z, w))v_j'(z)v_k'(w)\, dz\, d\zeta \\ &-\frac{1}{2\pi^2}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 T_{\alpha\beta}[v_\alpha'(z) - v_{\alpha'}'(z)] \times \\ &\qquad [v_\beta'(w) - v_{\beta'}'(w)]\wp_{jk}(v(\zeta, w))v_j'(\zeta)v_k'(w)dz\, d\zeta \\ &+\frac{1}{2\pi^2}\int_C\int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 T_{\alpha\lambda}T_{\beta\mu}[v_\alpha'(z) - v_{\alpha'}'(z)] \times \\ &[v_\beta'(\zeta) - v_{\beta'}'(\zeta)][v_\lambda'(w) - v_{\lambda'}'(w)][v_\mu'(w) - v_{\mu'}'(w)]dz\, d\zeta. \quad (28)\end{aligned}$$

We proved in "*Two Contours*," for the case specified in its title, that the variational condition,

$$\delta A(\mathbf{g}, R) = 0, \quad (29)$$

in the problem $A(\mathbf{g}, R) = \min$ implied the condition

$$\mathbf{F}'^2(w) = 0, \quad (30)$$

characteristic of a minimal surface. The essential point in the proof was the use of an identity equivalent to the combined addition theorems of the elliptic functions[17] ζ and $\wp$:

$$\wp(u + v) + \wp(u) + \wp(v) = [\zeta(u + v) - \zeta(u) - \zeta(v)]^2. \quad (31)$$

The identity actually used was the one resulting from this by the operator $\partial^2/\partial u\partial v$, being[18]

$$-\wp'(u+v)[\zeta(u)+\zeta(v)]+\wp(u+v)[\wp(u)+\wp(v)]+2\pi i\frac{\partial\wp(u+v)}{\partial\tau}=\wp(u)\wp(v). \quad (32)$$

Here, however, the functions ζ, $\wp$ were derived from the Jacobi function $\theta(u;\ \tau)$ with the characteristic $^1/_2$, $^1/_2$ $(=-\theta_1(u;\ \tau))$ instead of from the Weierstrass function σ, as is understood in (31)—i.e., in (32),

$$\zeta(u)=\frac{\partial\log\theta(u;\ \tau)}{\partial u},\qquad \wp(u)=-\frac{\partial^2\log\theta(u;\ \tau)}{\partial u^2}; \quad (33)$$

whereas in (31) ζ, $\wp$ are defined by the same formulas with $\sigma(u)$ of periods 1, τ in place of θ.

In the case of a general Riemann surface $\mathfrak{R}$ of any finite genus p, we stated and proved for the first time, in "*Minimal Surfaces . . .*,"[19] the following identity, generalizing (31):

$$\begin{aligned}[\wp_{jk}(v(z,w))+\wp_{jk}(v(z,\zeta))+\wp_{jk}(v(\zeta,w))]v_j'(w)v_k'(w)\\ =[\zeta_j(v(z,w))-\zeta_j(v(z,\zeta))-\zeta_j(v(\zeta,w))][\zeta_k(v(z,w))-\zeta_k(v(z,\zeta))\\ -\zeta_k(v(\zeta,w))]v_j'(w)v_k'(w)+R(w), \quad (34)\end{aligned}$$

where $R(w)$ denotes some (undetermined) function of w alone, rational on $\mathfrak{R}$, i.e., uniform and with only polar singularities. Here, similar to (33), the standard notation is used:

$$\zeta_j(u;\ \tau)=\frac{\partial\log\theta\ (u;\ \tau)}{\partial u_j},\quad \wp_{jk}(u;\ \tau)=-\frac{\partial^2\log\theta(u;\ \tau)}{\partial u_j\partial u_k}; \quad (35)$$

in a moment, we shall employ also

$$\wp_{jkl}(u;\ \tau)=-\frac{\partial^3\log\theta(u;\ \tau)}{\partial u_j\partial u_k\partial u_l}. \quad (36)$$

For the application to the Plateau problem, it was again necessary to apply to (34) the operator $\partial^2/\partial z\partial\zeta$, which eliminated the unknown function $R(w)$, and gave[20]

$$\begin{aligned}\Big\{&\wp_{jkl}(v(z,\zeta))[-\zeta_m(v(z,w))+\zeta_m(v(\zeta,w))]\\ &+\wp_{kl}(v(z,\zeta))\wp_{jm}(v(z,w))+\wp_{jm}(v(z,\zeta))\wp_{kl}(v(\zeta,w))\\ &+2\pi i\frac{\partial\wp_{jk}(v(z,\zeta))}{\partial\tau_{lm}}\Big\}v_l'(z)v_k'(\zeta)v_l'(w)v_m'(w)\\ &=\wp_{jl}(v(z,w))\wp_{km}(v(\zeta,w))v_j'(z)v_k'(\zeta)v_l'(w)v_m'(w). \quad (37)\end{aligned}$$

The summation convention as to repeated indices applies to j, k, l, m = 1, 2, . . . , p.

10. With the use of this identity, we proved in "*Minimal Surfaces . . .*" that also for the case of any number k of boundaries of R or of M, provided that the genus $h = 0$, the variational equation (29) of $A(\mathbf{g}, R)$ implies the condition (30) for a minimal surface.

One of the main features of the forthcoming Annals *paper is the easy extension of this proof to the case* $h > 0$.[21] This requires only a discussion of the complementary terms, as expressed by the formulas (27), (28), and is a perfectly routine matter, *not involving any such rather deep-lying identity as* (37). Indeed, the full details are presented in the next section, no. 11, of this note.

It thus becomes apparent that *all the essential difficulties of the more general case are already disposed of with the case* $h = 0$, provided that the number k of given contours is arbitrary. That this is so depends essentially on the fact that the genus p of the complete Riemann surface $\mathfrak{R}$, as expressed by the formula (3) $p = 2h + k - 1$, takes an arbitrary value as soon as k is arbitrary, even if h be restricted to the value zero. The new feature associated with $h > 0$ is that alter-symmetric circuits appear on $\mathfrak{R}$, which give rise to the complementary terms in Green's function, but, as we shall see in the next section, the effect of these is very easy to trace. The principal term of Green's function is the one involving Ω, and thereby the θ-function on $\mathfrak{R}$. It was the treatment of this term in application to the Plateau problem which required the identity (37).

11. We now give in detail the calculations involving the complementary terms of Green's function.

These produce the complementary terms of $A(\mathbf{g}, R)$, as given in (27). Putting aside the factor $1/4\pi$, we write for the kernel of the integral which expresses $\mathfrak{C}[A(\mathbf{g}, R)]$:

$$K = -T_{\alpha\beta}[v'_\alpha(z) - v'_{\alpha'}(z)][v'_\beta(\zeta) - v'_{\beta'}(\zeta)]. \tag{38}$$

As remarked in "*Minimal Surfaces . . .*" (p. 117), the Riemann surface $\mathfrak{R}$ can always be given a conformally equivalent form on which any preassigned point w of $\mathfrak{R}$ becomes a branch-point of first order (like that of $\sqrt{z}$). If this branch-point moves, then the normal integrals v of the first kind undergo the variation (loc. cit., formula 15)

$$\delta v_j(z) = -v_j'(w)v'_k(w)\zeta_k(v(z, w)). \tag{39}$$

This induces on the periods τ_{jk} of v the variation (loc. cit., formula 16)

$$\delta\tau_{jk} = 2\pi i v'_j(w)v'_k(w), \tag{40}$$

since the period of $\zeta_k(v(z, w))$ when z traverses a circuit B_m is known to be $-2\pi i\delta_{km}$.

We have then, by applying δ to (38),

$$\delta K = -T_{\alpha\beta}[\delta v'_\alpha(z) - \delta v'_{\alpha'}(z)][v'_\beta(\zeta) - v'_{\beta'}(\zeta)] - T_{\alpha\beta}[v'_\alpha(z) - v'_{\alpha'}(z)][\delta v'_\beta(\zeta) - \delta v'_{\beta'}(\zeta)] - \delta T_{\alpha\beta}[v'_\alpha(z) - v'_{\alpha'}(z)][v'_\beta(\zeta) - v'_{\beta'}(\zeta)]. \quad (41)$$

Differentiating the variational formula (39) as to z, we get

$$\delta v'_\alpha(z) - \delta v'_{\alpha'}(z) = [v'_\alpha(w) - v'_{\alpha'}(w)] \mathfrak{P}_{jk}(v(z, w))v'_j(z)v'_k(w). \quad (42)$$

Similarly,

$$\delta v'_\beta(\zeta) - \delta v'_{\beta'}(\zeta) = [v'_\beta(w) - v'_{\beta'}(w)] \mathfrak{P}_{jk}(v(\zeta, w))v_j'(\zeta)v'_k(w). \quad (43)$$

We recall the commutativity of differentiation and variation, a standard fact of the calculus of variations.

Apply to (15) the operator δ; then multiply by $T_{\beta\mu}$ and sum as to μ; this gives

$$\delta T_{\alpha\beta} = -T_{\alpha\lambda}T_{\beta\mu}\delta t_{\lambda\mu}. \quad (44)$$

By (11) and (39), we easily find

$$\delta t_{\lambda\mu} = [v'_\lambda(w) - v'_{\lambda'}(w)][v'_\mu(w) - v'_{\mu'}(w)]; \quad (45)$$

hence

$$\delta T_{\alpha\beta} = -T_{\alpha\lambda}T_{\beta\mu}[v'_\lambda(w) - v'_{\lambda'}(w)][v'_\mu(w) - v'_{\mu'}(w)]. \quad (46)$$

Combining (41), (42), (43) and (46), we obtain, with change of indices,

$$\delta K = -T_{\alpha\beta}[v'_\alpha(\zeta) - v'_{\alpha'}(\zeta)][v'_\beta(w) - v'_{\beta'}(w)]\mathfrak{P}_{jk}(v(z, w))v_j'(z)v'_k(w) - T_{\alpha\beta}[v'_\alpha(z) - v'_{\alpha'}(z)][v'_\beta(w) - v'_{\beta'}(w)]\mathfrak{P}_{jk}(v(\zeta, w))v_j'(\zeta)v'_k(w) + T_{\alpha\lambda}T_{\beta\mu}[v'_\alpha(z) - v'_{\alpha'}(z)][v'_\beta(\zeta) - v'_{\beta'}(\zeta)][v'_\lambda(w) - v'_{\lambda'}(w)][v'_\mu(w) - v'_{\mu'}(w)]. \quad (47)$$

By comparison with (28), this is seen to be *proportional exactly to the kernel of the integral which expresses* $\mathbf{C}[\mathbf{F}'^2(w)]$.

Thus it results that

$$\delta\mathbf{C}[A(\mathbf{g}, R)] = \frac{\pi}{2}\mathbf{C}[\mathbf{F}'^2(w)]. \quad (48)$$

But in "*Minimal Surfaces . . .*" we proved a precisely similar relation for the *principal terms* of $A(\mathbf{g}, R)$ and $\mathbf{F}'^2(w)$, as they appear in (25), (26), i.e.,

$$\delta A(\mathbf{g}, R) = \frac{\pi}{2}\mathbf{F}'^2(w), \qquad \textit{for } h = 0. \quad (49)$$

By adding this to the preceding formula, it follows that also *in the general case* $h > 0$, we have

$$\delta A(\mathbf{g}, R) = \frac{\pi}{2}\mathbf{F}'^2(w). \quad (50)$$

Consequently, in the general case likewise, the variational equation of $A(\mathbf{g}, R)$ *implies the condition* $\mathbf{F}'^2(w) = 0$, *which characterizes as a minimal surface the one defined by the harmonic vector* $\mathbf{H}(u, v)$ *on* R *with the boundary values* $\mathbf{g}$.

12. With this, the Plateau problem is solved in the generality stated at the beginning of this note. For the *attainment* of the minimum of $A(\mathbf{g}, R)$ has been established many times in the author's previous papers on the basis of the *compactness* of the argument range $[(\mathbf{g}, R)]$ and the *lower semi-continuity* of the functional A. We have only to prevent degeneration of the required minimal surface. This is done by means of certain simple sufficient conditions first given in the author's paper "*New Results*,"[22] which express that the lower bound of the areas of all *proper* surfaces of topological type r bounded by the given contours Γ is strictly less than the similar lower bound for *improper* surfaces of that type.[23]

In the following note an alternative method is presented of treating the general topological form of the Plateau problem, which uses Green's function in an intrinsic way, without employing for it any explicit formula—a circumstance which enables us to avoid many of the preceding calculations. This method proceeds in the same way in all cases, entirely regardless of the particular topological structure of the required minimal surface.

[1] The definition of r is the maximum number of circuits, no linear combination of which separates the surface. For a two-sided surface, $r = 2h$. For a one-sided surface, r may be odd or even; examples: Möbius surface with h handles, $r = 2h + 1$; Klein surface with h handles, $r = 2h + 2$. See Hilbert and Cohn-Vossen, *Anschauliche Geometrie*, 1932.

[2] J. Douglas, *Bull. Amer. Math. Soc.*, **36**, 50 (1930).

[3] In the following publications, respectively: *Trans. Amer. Math. Soc.*, **33**, 263–321 (1931); *Journ. Math. Phys.*, **10**, 310–359 (1931); *Trans. Amer. Math. Soc.*, **34**, 731–756 (1932). The second of these will be referred to as "*Two Contours*."

[4] Loc. cit., p. 324. See Formulas (23), (24) of the present note.

[5] J. Douglas, "*Some New Results in the Problem of Plateau*," *Journ. Math. Phys.*, **15**, 55–64 (Feb., 1936); "*Minimal Surfaces of General Topological Structure with any Finite Number of Assigned Boundaries*," *Ibid.*, **15**, 105–123 (June, 1936). These papers will be referred to respectively as "*New Results*" and "*Minimal Surfaces . . .*"

[6] These PROCEEDINGS, **22**, 367–372 (June, 1936).

[7] R. Courant, *Ann. Math.*, **38**, 679–724 (1937).

[8] *Rendiconti Lincei*, **24**, 333–339, 393–398 (Nov., Dec., 1936).

[9] Specifically, the new items included in the present note are formulas (2), (27), (28) and §11.

[10] As far as I know, an explicit formula such as (2) has not hitherto been given for the Green's function of a general Riemann surface. (1) was given in "*Minimal Surfaces...*" as formula (2) (equivalent to formulas (1) and (4) of the present note).

[11] For abelian integrals and algebraic geometry, see, throughout this note, E. Picard, *Traité d'Analyse*, **2**, chaps. 14–16 (1925), and Severi-Löffler, *Vorlesungen über Algebraische Geometrie*, 1921.

[12] See the definition, immediately after formula (8).

[13] See the author's previously cited paper in *Trans. Amer. Math. Soc.*, **34**, 731–756 (1932). By means of the relation just described between one- and two-sided surfaces,

the case $r = 1$ of a Möbius surface with any finite number k of boundaries can be referred to the case $h = 0$, where there are $2k$ boundaries, symmetric in pairs. Accordingly, the case $r = 1$ can likewise be treated with the simpler formula (1) for Green's function.

[14] See H. F. Baker, "*Abel's Theorem and . . . Theta Functions*," 1897, chapter X.

[15] In the notation of symmetric indices j, j' which we introduce immediately hereafter, we suppose also $\alpha_j = \alpha_{j'}$, $\beta_j = \beta_{j'}$.

The α, β which denote the characteristic of the θ-series have, of course, nothing to do with the summation indices of formula (2). For the rest of this note, α, β will be used exclusively in the latter meaning.

[16] Systematically, the letters **H**, **g**, **F** in bold-faced type will denote *vectors* with n components.

[17] As notation, the function ζ is easily distinguished from the argument ζ which appears in (23) et seq.

[18] Equivalent to (7.1) of "*Two Contours*."

[19] Loc. cit., formula (41).

[20] "*Minimal Surfaces . . .*," formula (33).

[21] Loc. cit., §12.

[22] Theorems I, II, and footnote 9; see as well the similarly numbered theorems of the forthcoming *Annals* paper.

[23] That is, surfaces S bounded by Γ which are expressible in the form $S = S_1 + S_2 + \ldots + S_m$ where the respective characteristics obey the condition $r_1 + r_2 + \ldots + r_m \leqq r$ and either $m > 1$ or the relation $<$ holds.

GREEN'S FUNCTION AND THE PROBLEM OF PLATEAU

BY JESSE DOUGLAS

BROOKLYN, N. Y.

Communicated July 1, 1938

1. The general topological form of the problem of Plateau requires the determination of a minimal surface M of any given finite genus h or characteristic r, either character of orientability (one- or two-sided), and with any finite number k of boundaries Γ, assigned in form and position in n-dimensional euclidean space.[1]

The author has based his solution of this problem[2] on the functional

$$A(\mathbf{g}, R) = D(\mathbf{H}) = \int\int_R \frac{1}{2}(E + G)\, du\, dv = \frac{1}{2}\int\int_R (\mathbf{H}_u^2 + \mathbf{H}_v^2)\, du\, dv. \tag{1}$$

Here R denotes any Riemann surface having the topological form prescribed for the minimal surface M; $\mathbf{g}$ denotes any parametric representation of the contours Γ on the boundaries C of R. $\mathbf{H}(u, v)$ is the harmonic vector function on R with the values $\mathbf{g}$ on C; E, F, G denote the first fundamental quantities of the surface $\mathbf{x} = \mathbf{H}(u, v)$; $\mathbf{g}, \mathbf{H}, \mathbf{x}$ are, throughout, *vectors*

of n components; D, applied to a vector, denotes one-half the sum of the Dirichlet integrals of its components.

A minimal surface M is one definable as a *harmonic* and *conformal* image of a Riemann surface R, i.e., in the form

$$\mathbf{x} = \mathbf{H}(u, v), \tag{2}$$

with

$$E = G, \qquad F = 0. \tag{3}$$

These express the variational condition in the problem of the surface of least area with given boundaries.

To include one-sided minimal surfaces $\overline{M}$ in our theory, we employ a two-sided covering surface in two-one point correspondence with $\overline{M}$.[3]

2. *Riemann Surfaces.*—The Riemann surface R may always be considered as one of the conjugate halves (semi Riemann surface) of a *symmetric* Riemann surface $\mathfrak{R}$. The symmetric property of $\mathfrak{R}$ consists in the existence of an inversely conformal transformation T which associates the points of $\mathfrak{R}$ in pairs P, $\overline{P}$, called conjugate or symmetric. $\mathfrak{R}$ may always be represented as the Riemann surface of a real algebraic curve $\mathfrak{A}: P(x, y) = 0$ (real coefficients), and T may then be interpreted as the interchange of conjugate complex points (x, y), $(\bar{x}, \bar{y})$. The points fixed under T constitute the real branches C of $\mathfrak{A}$. $\mathfrak{A}$ is subject to any real birational transformation without essential change in $\mathfrak{R}$; i.e., $\mathfrak{R}$ remains conformal to itself.

The complete complex manifold $\mathfrak{A}$ may be represented by writing in $P(x, y) = 0$,

$$x = x_1 + ix_2, \qquad y = y_1 + iy_2, \tag{4}$$

giving

$$P(x_1 + ix_2, y_1 + iy_2) = 0, \tag{5}$$

which, by separation of real and imaginary parts, gives two equations

$$P_1(x_1, x_2, y_1, y_2) = 0, \qquad P_2(x_1, x_2, y_1, y_2) = 0. \tag{6}$$

These represent in the ordinary four-dimensional space (x_1, x_2, y_1, y_2) a two-dimensional manifold $\mathfrak{S}$, which may be taken as one of the forms of the Riemann surface of $\mathfrak{A}$. The *section* of $\mathfrak{S}$ by the real (x_1, y_1)-plane gives the real branches of $\mathfrak{A}$. The *orthogonal projection* of $\mathfrak{S}$ on the complex x- and y-planes gives the Riemann surfaces $\mathfrak{R}_x$, $\mathfrak{R}_y$ in the standard form of many-sheeted surfaces over these planes. $\mathfrak{S}$, $\mathfrak{R}_x$, $\mathfrak{R}_y$ are all in conformal correspondence by this projection, and are therefore equivalent for all our purposes.

If we suppose that the real branches of $\mathfrak{A}$ separate the Riemann surface $\mathfrak{S}$, or $\mathfrak{R} = \mathfrak{R}_x$ or $\mathfrak{R}_y$,[4] then either of the conjugate halves may be taken as R. The boundaries of R are then precisely these real branches C; and R,

previously called a semi Riemann surface, may be referred to simply as a Riemann surface, except that it has boundaries while the complete Riemann surface $\mathfrak{R}$ is closed.

3. In the solutions previously given by the author for the general topological form of the Plateau problem, the analytic procedure was based on an explicit formula for the Green's function of R in terms of θ-functions and abelian integrals on the complete Riemann surface $\mathfrak{R}$.[5]

The Green's function $G(P_1, P_2)$ of R is uniquely defined by the following properties. As a function of the point P_1, $G(P_1, P_2)$ is: (i) uniform on R, (ii) harmonic on R, (iii) logarithmically singular for $P_1 = P_2$, (iv) equal to zero when P_1 is on the boundary C of R. Property (iii) means that for P_1 in the vicinity of P_2, we have

$$G(P_1, P_2) = -\log P_1P_2 + G_1(P_1, P_2) \tag{7}$$

where $G_1(P_1, P_2)$ is regular for $P_1 = P_2$. We may recall also the well-known symmetry property

$$G(P_1, P_2) = G(P_2, P_1). \tag{8}$$

The purpose of this note is to present the essential features of a simpler method of treatment, which uses Green's function in an intrinsic way, without employing for it any explicit expression. This method proceeds in the same way in all cases, regardless of the particular topological form of R or M.

A detailed presentation will appear elsewhere under the same title.

4. Equivalent to the defining formulas (2), (3) of a minimal surface M are the following:

$$\mathbf{x} = \mathbf{H}(Q), \tag{9}$$

$$\frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta} = 0.[6] \tag{10}$$

Here Q denotes an arbitary interior point of R, and ξ, η are any two perpendicular directions on R at Q.

For (10) expresses that any two perpendicular directions on R are converted into perpendicular directions on M, which is sufficient to secure the *conformal* nature of the correspondence between the two surfaces. Indeed, the conditions (3) of conformality are merely two particular cases of (10), namely, for ξ, η coincident with the u, v directions, and with their angle-bisectors.

M will be bounded by given contours Γ if the values of $\mathbf{H}(Q)$ on the boundaries C of R, i.e.,

$$\mathbf{x} = \mathbf{g}(P), \tag{11}$$

form a parametric representation of Γ.

5. It will convey a sufficiently typical impression of the contents of our detailed paper if we simply reproduce its three fundamental formulas, and quote its main theorem. The latter provides an explicit construction, in both geometric and analytic form, of a special variation of the Riemann surface R, which realizes the variational formula of Green's function given hereafter as (16).

We imagine that the variation of R to R_ϵ takes place by simultaneous variation of the individual points: P_1 to $P_1(\epsilon)$, P_2 to $P_2(\epsilon)$, etc. Then Green's function $G(P_1, P_2)$, which depends on the form of the Riemann surface R_ϵ and the position of the points $P_1(\epsilon)$, $P_2(\epsilon)$, becomes, for given points P_1, P_2 on R, a function of ϵ:

$$G_\epsilon(P_1(\epsilon), P_2(\epsilon)) = G(\epsilon;\ P_1, P_2). \tag{12}$$

We define the variation of $G(P_1, P_2)$ by the usual formula[7]

$$\delta G(P_1, P_2) = \frac{\partial}{\partial \epsilon} G(\epsilon;\ P_1, P_2)\Big|_{\epsilon=0}. \tag{13}$$

6. *Formulas.*—The three basic formulas referred to are the following:

$$A(\mathfrak{g}, R) = \frac{1}{8\pi}\int_C\int_C [\mathfrak{g}(P_1) - \mathfrak{g}(P_2)]^2 \frac{\partial^2 G(P_1, P_2)}{\partial n_1\, \partial n_2}\, ds_1\, ds_2. \tag{14}$$

$$\frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta} = -\frac{1}{16\pi^2}\int_C\int_C [\mathfrak{g}(P_1) - \mathfrak{g}(P_2)]^2$$

$$\frac{\partial^2}{\partial n_1\, \partial n_2}\left\{\frac{\partial G(P_1, Q)}{\partial \xi}\frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta}\frac{\partial G(P_2, Q)}{\partial \xi}\right\} ds_1\, ds_2. \tag{15}$$

$$\delta G(P_1, P_2) = \frac{\partial G(P_1, Q)}{\partial \xi}\ \frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta}\ \frac{\partial G(P_2, Q)}{\partial \xi}. \tag{16}$$

Here $\partial/\partial n_1$, $\partial/\partial n_2$ denote differentiation in the direction of the interior normal to C at P_1, P_2, respectively.

The bearing of these formulas on the Plateau problem is apparent. For by combining them, we evidently get

$$\delta A(\mathfrak{g}, R) = -2\pi \frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta}. \tag{17}$$

In this, we use the readily established fact of the commutativity of the operators δ and $ds_1\, ds_2\, \partial^2/\partial n_1 \partial n_2$.

It follows that the variational condition

$$\delta A(\mathfrak{g}, R) = 0, \tag{18}$$

associated with our fundamental minimum principle

$$A(\mathfrak{g}, R) = \min, \tag{19}$$

is equivalent exactly to the condition (10) for a minimal surface.

The variational formula (16) is thus seen to serve the same purpose, in the present method of solution, as was served in our previous method by the basic identity in θ-functions.[8] For the purpose of that identity was precisely to bring about the equivalence of the variational condition on the functional $A(\mathfrak{g}, R)$ to the condition for a minimal surface.

In connection with the formula (14), we may remark that in the simplest case, where R is the unit circle and C its circumference, on which P_1, P_2 with polar angles θ, φ are any two points, we have by standard formulas,

$$\frac{\partial^2 G(P_1, P_2)}{\partial n_1, \partial n_2} = \frac{1}{2\sin^2 \dfrac{\theta - \varphi}{2}}. \tag{20}$$

The general formula (14) then reverts to the particular form[9]

$$A(\mathfrak{g}) = \frac{1}{4\pi}\int_C \int_C \frac{[\mathfrak{g}(\theta) - \mathfrak{g}(\varphi)]^2}{4\sin^2 \dfrac{\theta - \varphi}{2}}\, d\theta\, d\varphi, \tag{21}$$

on which we based our original solution for a single given contour and a simply-connected minimal surface.

7. We may now quote our main theorem, and then conclude with some explanatory figures and remarks.

VARIATIONAL THEOREM CONCERNING GREEN'S FUNCTION

THEOREM. *Let* $\mathfrak{A}$ *denote any real algebraic curve, on whose Riemann surface,* $\mathfrak{S}$ *or* $\mathfrak{R}$, *the points* Q, $\bar{\text{Q}}$ *are any two conjugate imaginary points.*

Let the tangent lines t, $\bar{\text{t}}$ *to* $\mathfrak{A}$ *at* Q, $\bar{\text{Q}}$ *intersect in the real point* O. *Choose a reference triangle with one vertex at* O; *then in homogeneous line coördinates* u, v, w, *the equation of* $\mathfrak{A}$ *will evidently have the form*

$$\mathfrak{A}: \qquad (au^2 + buv + cv^2)\, K(u, v) + w\, L(u, v, w) = 0,$$

where

$$t, \bar{t}: \qquad au^2 + buv + cv^2 = 0$$

represents the conjugate imaginary tangents t, $\bar{\text{t}}$. K *and* L *are homogeneous polynomials.*

Construct now the family of curves with parameter ϵ,

$$\mathfrak{A}_\epsilon: \ [(a + a'\epsilon)u^2 + (b + b'\epsilon)uv + (c + c'\epsilon)v^2]\, K(u, v) + wL(u, v, w) = 0,$$

where a′, b′, c′ *are fixed but arbitrary real coefficients.*

Then by rectilinear projection $\mathfrak{P}$ *from* O, *the Riemann surfaces* $\mathfrak{S}$, $\mathfrak{S}_\epsilon$ *of* $\mathfrak{A}$ *and* $\mathfrak{A}_\epsilon$ *are set into one-one continuous and conformal correspondence:* P *to* P(ϵ), *except in the immediate vicinity of* $Q, \overline{Q}$. *In particular, the real branches* C, C_ϵ *of* $\mathfrak{A}$ *and* $\mathfrak{A}_\epsilon$ *are thereby set into one-one continuous correspondence. This depends on the circumstance that all the real tangents from* O *to* $\mathfrak{A}_\epsilon$— *which are defined by the real factors of the equation*

$$K(u, v) = 0$$

—*remain invariant, since this equation is independent of* ϵ.

In fact, for the same reason, all the tangents, real and imaginary, from O *to* $\mathfrak{A}_\epsilon$, *except* t_ϵ, $\bar{t}_\epsilon$, *remain fixed. These, however, vary with* ϵ; *and* t_ϵ, *for instance, intersects the Riemann surface* $\mathfrak{S}$ *of* $\mathfrak{A}$ *in two points near to* Q,[10] *which, as* ϵ *passes through the value zero from positve to negative, always enter* Q *from two opposite directions* α, α' *and leave in the perpendicular opposite directions* β, β'. *Let the angle-bisectors of* $\alpha\beta$, $\alpha\beta'$ *be the perpendicular directions* ξ, η.

Denote by $G(P_1, P_2)$ *the Green's function of either conjugate semi-surface* R *of the symmetric Riemann surface* $\mathfrak{S}$.[11] *Then, under the precedingly described variation of the form of* $\mathfrak{S}$ *and the position of the points* P_1, P_2, *we have for Green's function the variational formula*

$$(V): \quad \delta G(P_1, P_2) = \frac{\partial G(P_1, Q)}{\partial \xi} \frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta} \frac{\partial G(P_2, Q)}{\partial \xi}$$

(*apart from an inessential numerical factor*).

Finally, the directions ξ, η *can be made to coincide with any preassigned perpendicular directions at* Q, *by proper choice of* a′, b′, c′.

This theorem seems of particular interest for its interplay of fundamental analytic and geometric entities, as well as for its direct application to the Plateau problem.

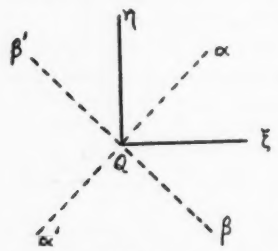

FIGURE 1

8. Figure 1 shows the directions α, α'; β, β'; ξ, η; referred to in the preceding statement.

Figure 2 illustrates the real branches of the curve $\mathfrak{A}$ drawn full, and of the varied curve $\mathfrak{A}_\epsilon$ drawn dotted. The real tangents from O are indicated by full drawn lines. One other projecting line from O is drawn dotted, and upon it corresponding points P, P_ϵ are indicated.

It is evident how the fact that $\mathfrak{A}_\epsilon$ remains always tangent to the *same real lines* through O conditions the one-to-one nature of the correspondence between the *real branches* of $\mathfrak{A}$ and $\mathfrak{A}_\epsilon$ established by the projection from O.[12] For, otherwise, either $\mathfrak{A}$ or $\mathfrak{A}_\epsilon$ would protrude outside one of the real

tangents to the other from O, and then the protruding arc, say of $\mathbf{A}_\epsilon$, could have no corresponding real arc on $\mathbf{A}$.

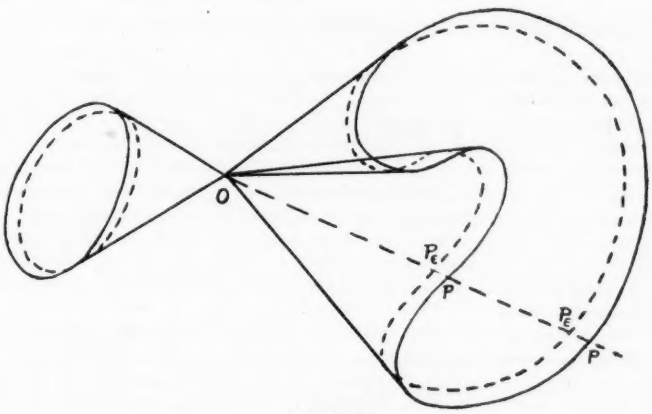

FIGURE 2

REFERENCES

J. Douglas:

1. "Some New Results in the Problem of Plateau," *Jour. Math. Phys.*, **15**, 55–64 (1936).
2. "Minimal Surfaces of General Topological Structure with any Finite Number of Assigned Boundaries," *Ibid.*, **15**, 105–123 (1936).
3. "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, 263–321 (1931).
4. "The Problem of Plateau," *Bull. Amer. Math. Soc.*, **39**, 227–251 (1933).
5. "The Problem of Plateau for Two Contours," *Jour. Math. Phys.*, **10**, 310–359 (1931).
6. "One-Sided Minimal Surfaces with a Given Boundary," *Trans. Amer. Math. Soc.*, **34**, 731–756 (1932).
7. "Crescent-Shaped Minimal Surfaces," these PROCEEDINGS, **19**, 192–199 (1933).
8. "Minimal Surfaces of Higher Topological Structure," *Ann. Math.* (in press).
9. The foregoing note in these PROCEEDINGS, with the same title as [8].

[1] This general form of the Plateau problem was first formulated by the author in *Bull. Amer. Math. Soc.*, **36**, 50 (1930).

[2] In [1], [2], [8], [9] of the list of references at the end of this paper, citations from which will be made by numbers in square brackets. The other papers in the list deal with important particular cases leading up to the general one: one contour [3], [4]; two non-intersecting contours [5]; Möbius strip [6]; two contours intersecting in a single point [7]. In these particular cases, except for [6], where the characteristic r is unity, the required minimal surfaces were required to be of genus $h = 0$.

[3] Cf. [6], p. 734; [1], p. 61.

[4] Symmetric Riemann surface of the *first kind;* see F. Klein, *Über Riemanns Theorie der Algebraischen Funktionen und ihrer Integrale*, 1882.

[5] [8], formula (7.17); [9], formula (2); for the case $h = 0$, [2], formula (2).

[6] The dot denotes the scalar product of vectors. The exponent two (formulas 14, 15) will denote the scalar product of a vector by itself, i.e., the sum of the squares of its components.

[7] Except that we interpret δ as a *derivative*, rather than a differential, as is more customary.

[8] Formulas (41) and (33) of [2], reproduced as (34) and (37) of the foregoing note in the present issue of these PROCEEDINGS. This is for a general topological form of the Riemann surface $\mathfrak{R}$. With increasing complexity of $\mathfrak{R}$, the appropriate identity involved successively: algebraic functions, trigonometric functions, elliptic functions, θ-functions. See, respectively [4], p. 243; [7], formula (5.4); [5], formula (7.1); [2], formulas (41), (33). In [7], the functions actually appearing are hyperbolic, due to the rotation through a right angle of the parallel strip representing the Riemann surface R.

[9] Given in [3], and in preceding abstracts in *Bull. Amer. Math. Soc.*, **36**, 50 (1930).

[10] We suppose that the contact of the tangents $t, \bar{t}$ to $\mathbf{A}$ at $Q, \bar{Q}$, respectively, is ordinary two-point contact. Higher contact can always be avoided by a preliminary birational transformation.

[11] Or its conformally equivalent orthogonal projections $\mathfrak{R} = \mathfrak{R}_x$ or $\mathfrak{R}_y$ (see art. 2). The notation is supposed arranged so that the semi-surface R contains the point Q.

[12] We may again remark that this one-to-one character breaks down in the vicinity of the points of contact $Q, \bar{Q}$ of the tangents $t, \bar{t}$. Otherwise, it extends beyond the real branches to the rest of the Riemann surfaces involved, with deletion of the stated neighborhoods.

THE MOST GENERAL FORM OF THE PROBLEM OF PLATEAU

BY JESSE DOUGLAS

BROOKLYN, New York

Communicated June 28, 1938

1. The method of the preceding note—cited hereafter as Note II—applies with practically no modification to the following "most general formulation" of the problem of Plateau, which, I believe, is given here explicitly for the first time.

PROBLEM P. *Given any riemannian manifold* R *in the most general sense of the term, i.e., any two-dimensional connected topological variety for which there is defined in the neighborhood of each point a local conformal representation on a circle.*[1] R *may then have any finite or infinite number of boundaries, and any topological structure whatever, i.e., any finite or infinite type of connectivity. It may also have either character of orientability, i.e., one- or two-sidedness.*[2]

Given also any point-set Γ *in* n*-dimensional euclidean space which is a topological image of the total boundary* C *of* R. Γ *may consist of any finite or infinite number of Jordan curves, together with their limit points; or it may be*

some more general type of point-set. A definite sense of description is associated with each component of Γ, *carried over from* C, *which may be supposed to be oriented so that* R *is on the left.*

To determine the existence of a minimal surface M *topologically equivalent to* R *and bounded by* Γ.

2. The exact analytic meaning of the requirement expressed in the last sentence is the same as in art. 4 of Note II.

If R has only a finite number k of boundaries and finite topological characteristic r, then we have the problem dealt with in Note II. This most important case of Problem P we shall call Problem P_0.

In Problem P_0, R is the Riemann semi-surface of a real *algebraic* curve; in Problem P, R is the Riemann semi-surface of a real *analytic* curve $\mathfrak{A}$. This means the manifold that is obtained from the Riemann surface $\mathfrak{R}$ associated with the equation $y = f(x)$ of the analytic curve $\mathfrak{A}$ by combining into a single geometric element each pair of conjugate complex points $(x, y), (\bar{x}, \bar{y})$ of $\mathfrak{A}$. That $\mathfrak{A}$ is *real* means that the power-series elements of the analytic function $y = f(x)$ occur in conjugate complex pairs, i.e., with conjugate complex centers x_0, $\bar{x}_0$ and coefficients a_m, $\bar{a}_m$.

The *boundary* C of R consists of the *real branches* of $\mathfrak{A}$, and R is two- or one-sided according as C does or does not separate $\mathfrak{R}$.

Thus, if $\mathfrak{A}$ is the curve

$$y^2 = \sin x \tag{1}$$

we have the problem of a minimal surface M topologically equivalent to a sphere with an infinite number of perforations that converge to a single point p, and where the prescribed boundary Γ of M consists of an infinite number of Jordan curves in space which converge to a single point P.

If, as another example, $\mathfrak{A}$ is

$$y^2 = (x^2 + 1) \sin x, \tag{2}$$

then the form required for M is that of a Möbius surface with an infinite number of boundaries, which have assigned positions Γ in space, of the same type of distribution as before.

If $\mathfrak{A}$ is

$$y^2 = \sin x \cosh x, \tag{3}$$

then the form of M is that of a one-sided surface with an infinite number of handles or branch-cuts converging to a point and an infinite number of given boundaries Γ converging to a point.

3. We introduce the following notation and definitions, and quote also the simple preliminary theorems (7°), (12°).

(1°) $\mathfrak{g}$ shall denote any representation of the given complete boundary Γ as topological image of the total boundary C of R.

(2°) *Definition.*

$$A(\mathfrak{g}, R) = \frac{1}{2}\int\int_R (\mathbf{H}_u^2 + \mathbf{H}_v^2)\, du\, dv, \tag{4}$$

where $\mathbf{H}(u, v)$ is the harmonic vector function on R whose boundary values are $\mathfrak{g}$.

(3°) $\mathfrak{T}$ shall denote the class of all riemannian manifolds topologically equivalent to R.

(4°) $\mathfrak{T}'$ shall denote all riemannian manifolds R' which are the limit of a sequence of manifolds R_m of $\mathfrak{T}$, without belonging to $\mathfrak{T}$ themselves.

Any such manifold R' either consists of a finite or infinite number of separate parts, or, if consisting of a single component, is of lower topological type than R. If the equation of the analytic curve $\mathbf{A}'$ corresponding to R' is $F(x, y) = 0$, then, accordingly, either $F(x, y)$ is reducible, separating into a finite or infinite number of factors, or else it acquires new conjugate complex multiple points by coalescence of branch-points of R.

$\mathfrak{T}'$ may be regarded as the "frontier" of $\mathfrak{T}$ in the "space" of all riemannian manifolds.

(5°) *Definition.*

$$d(\Gamma, \mathfrak{T}) = \min A(\mathfrak{g}, R), \tag{5}$$

for all riemannian manifolds R belonging to $\mathfrak{T}$, and all topological representations $\mathfrak{g}$ of Γ as image of the boundary C of R. Throughout, we understand "min" in the sense of *lower bound*, without prejudice of the question as to whether the minimum is attained or not.

The value of d $(\Gamma, \mathfrak{T})$ *may be either a finite positive number, or* $+\infty$, depending on the case.

(6°) *Definition.*

$$d(\Gamma, \mathfrak{T}') = \min A(\mathfrak{g}, R'), \tag{6}$$

for all manifolds R' of $\mathfrak{T}'$, and all topological representations $\mathfrak{g}$ of Γ as image of the total boundary C of all components of R'.

(7°) THEOREM. *In every case,*

$$d(\Gamma, \mathfrak{T}) \leqq d(\Gamma, \mathfrak{T}'). \tag{7}$$

(8°) *Definition. For* $d(\Gamma, \mathfrak{T})$ *finite,*

$$e(\Gamma, \mathfrak{T}) = d(\Gamma, \mathfrak{T}') - d(\Gamma, \mathfrak{T}). \tag{8}$$

By the relation (7),

$$e(\Gamma, \mathfrak{T}) \geqq 0. \tag{9}$$

(9°) *Definition. Whether* $\mathrm{d}(\Gamma, \mathfrak{T})$ *is finite or infinite,*

$$\bar{e}(\Gamma, \mathfrak{T}) = \max \overline{\lim_{m \to \infty}}\, e\,(\Gamma_m, \mathfrak{T}_m), \tag{10}$$

where Γ_m tends to Γ and $\mathfrak{T}_m$ to $\mathfrak{T}$ in such a way that $d(\Gamma_m, \mathfrak{T}_m)$ is finite for $m = 1, 2, 3, \ldots$. Sequences of this kind always exist for any given $(\Gamma, \mathfrak{T})$, since we may take $\mathfrak{T}_m$ to have always a finite number of boundaries and finite genus, and Γ_m to consist of a finite number of polygons. The maximum in (10) is with respect to all such sequences. "$\overline{\lim}$" denotes the superior limit.

Obviously, by (9), we have in every case,

$$\bar{e}(\Gamma, \mathfrak{T}) \geqq 0. \tag{11}$$

(10°) *Definition.*

$$a(\Gamma, \mathfrak{T}) = \min \mathfrak{A}(S) \tag{12}$$

where $\mathfrak{A}(S)$ denotes the area of S, which ranges over all surfaces of topological type $\mathfrak{T}$ bounded by Γ.

(11°) *Definition.*

$$a(\Gamma, \mathfrak{T}') = \min \mathfrak{A}(S') \tag{13}$$

where the range of S' consists of all surfaces of type $\mathfrak{T}'$ bounded by Γ.

(12°) THEOREM.

$$a(\Gamma, \mathfrak{T}) = d(\Gamma, \mathfrak{T}), \qquad \textit{also } a(\Gamma, \mathfrak{T}') = d(\Gamma, \mathfrak{T}'). \tag{14}$$

4. We are now in a position to state our main theorems.

THEOREM 1. *If* $\mathrm{d}(\Gamma, \mathfrak{T})$ *is finite, and we have the strict form of inequality*[3]

$$d(\Gamma, \mathfrak{T}) < d(\Gamma, \mathfrak{T}'), \tag{15}$$

then a minimal surface M *exists, of topological type* T, *bounded by* Γ.

The area of M *is the least possible for its prescribed topological type and boundaries:*

$$\mathfrak{A}(M) = a(\Gamma, \mathfrak{T}). \tag{16}$$

THEOREM 2. *Whether* $\mathrm{d}(\Gamma, \mathrm{T})$ *is finite or infinite, if* $\bar{e}(\Gamma, \mathrm{T})$*—always non-negative*[4]*—is actually positive:*

$$\bar{e}(\Gamma, \mathfrak{T}) > 0, \tag{17}$$

then a minimal surface M *exists, of topological type* $\mathfrak{T}$, *bounded by* Γ.

If $\mathrm{d}(\Gamma, \mathfrak{T}) = \mathrm{a}(\Gamma, \mathfrak{T})$ *is infinite, then so is the area of* M, *but every completely interior sub-region* M_1 *of* M *has an area which is finite and a minimum for its own topological type* $\mathfrak{T}_1$ *and boundaries* Γ_1.

5. The proof of these theorems is practically the same as for the finite or algebraic case of Problem $\mathrm{P}_\circ$, requiring hardly more than the appropriate changes of wording. The same basic formulas (14), (15), (16) of Note II, are applied.

The important observation is that *the Variational Theorem of Note II (art. 7) continues to hold,* mutatis mutandis, *for all real analytic as well as algebraic curves* $\mathbf{A}$.

We employ non-homogeneous line coördinates, namely the coefficients u, v in

$$y = ux + v. \tag{18}$$

We also take the point O in the statement of the Variational Theorem for origin, so that its line equation is $v = 0$. Then the line equation of the real analytic curve $\mathbf{A}$ will have the form

$$(au^2 + bu + c)\,K(u) + vL(u, v) = 0, \tag{19}$$

where K, L are now *any analytic functions*, instead of polynomials, as in Note II. The quadratic factor

$$au^2 + bu + c = 0 \tag{20}$$

represents the two conjugate imaginary tangents t, $\bar{t}$. For the varied equation, we have

$$[(a + a'\epsilon)u^2 + (b + b'\epsilon)u + (c + c'\epsilon)]\,K(u) + vL(u, v) = 0, \tag{21}$$

quite analogous to Note II.

6. The detailed version of the present note will be published elsewhere under the same title.

[1] In accordance with certain two simple postulates, which express essentially that the angle-metric induced on R by this conformal representation of its neighborhoods on a circle is self-consistent whenever two neighborhoods overlap. Cf. H. Weyl, *Die Idee der Riemannschen Fläche*, 36 (1913); L. Ahlfors, "Geometrie der Riemannschen Flächen," *Proc. Oslo Congress*, **1**, 239–248 (1936); S. Stoilow, "Sur la définition des surfaces de Riemann," *Ibid.*, **2**, 143; S. Stoilow, *Principes topologiques des fonctions analytiques*, (1937).

[2] This does not contradict Weyl, loc. cit., p. 66: "*Jede Riemannsche Fläche ist zweiseitig.*" For this theorem is associated with the idea of an analytic function $f(t)$ on the Riemann surface, t being sharply distinguished from its conjugate $\bar{t}$, whereas our geometric form of definition obviously permits one-sidedness of R. In that case, we employ in our analysis a covering surface R_1, which is two-sided and in two-one correspondence with R; cf. Weyl, loc. cit., last lines of p. 61.

[3] See (7).

[4] See (11).

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL and of other American investigators. The PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of American research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text-figures of simple character, though more elaborate illustrations may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

REFERENCES to literature, numbered consecutively, will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some detail and in general in accordance with the standard adopted for the Subject Catalogue of the International Catalogue of Scientific Literature, viz., name of author, with initials following (ordinarily omitting title of paper), abbreviated name of Journal, with place of publication, series (if any), volume, inclusive pages, year. For example: Montgomery, T. H., *J. Morph., Boston*, 22, 731–815 (1911); or, Wheeler, W. M., *Konigsburg, Schr. physik., Ges.*, 55, 1–142 (1914).

PAPERS by members of the ACADEMY may be sent to Edwin Bidwell Wilson, Managing Editor, Harvard School of Public Health, Boston 17, Mass. Papers by non-members of the ACADEMY or COUNCIL should be submitted through some member.

PROOF will not ordinarily be sent; if an author asks for proof, it will be sent with the understanding that charges for his corrections shall be billed to him. Authors are therefore requested to make final revisions on the typewritten manuscripts. The editors cannot undertake to do more than correct obvious minor errors.

REPRINTS should be ordered at the time of submission of manuscript. They will be furnished to authors at cost, approximately as follows:

SCHEDULE OF RATES FOR REPRINTS, CARRIAGE CHARGES EXTRA

No. copies	50	100	150	200	250
4 pp.	$2.10	$2.40	$2.70	$3.00	$3.30
8 pp.	4.20	4.80	5.40	6.00	6.60
Covers	1.60	2.05	2.50	2.95	3.40

CONTENTS

Page

BOTANY.—THE ORIGIN OF MAIZE . . *By P. C. Mangelsdorf and R. G. Reeves* 303

PHYSIOLOGY.—CALCULATIONS OF BIOELECTRIC POTENTIALS. III. VARIATION IN PARTITION COEFFICIENTS AND ION MOBILITIES . . . . . *By S. E. Hill and W. J. V. Osterhout* 312

PHYSICS.—RAYLEIGH WAVES . . . . . *By H. Bateman* 315

PHYSICS.—COULOMB'S FUNCTION . . . . . *By H. Bateman* 321

MATHEMATICS.—FUNCTIONAL TOPOLOGY AND ABSTRACT VARIATIONAL THEORY . *By Marston Morse* 326

MATHEMATICS.—ON CRITERIA CONCERNING SINGULAR INTEGERS IN CYCLOTOMIC FIELDS. . . . . *By H. S. Vandiver* 330

MATHEMATICS.—GROUPS OF DEGREE n INVOLVING ONLY SUBSTITUTIONS OF LOWER DEGREES . . . . . *By G. A. Miller* 333

MATHEMATICS.—ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE. . . . . *By J. L. Walsh and W. Seidel* 337

MATHEMATICS.—DIFFERENTIAL CALCULUS IN LINEAR TOPOLOGICAL SPACES. . *By A. D. Michal* 340

MATHEMATICS.—MINIMAL SURFACES OF HIGHER TOPOLOGICAL STRUCTURE . . *By Jesse Douglas* 343

MATHEMATICS.—GREEN'S FUNCTION AND THE PROBLEM OF PLATEAU . . . . *By Jesse Douglas* 353

MATHEMATICS.—THE MOST GENERAL FORM OF THE PROBLEM OF PLATEAU . . *By Jesse Douglas* 360

ge
03
12
15
21
26
30
33
37
40
43
3
0